PROJECTIVE GEOMETRY

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PROJECTIVE GEOMETRY

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PREFACE

THIS volume is a sequel to the author's *Modern Geometry* which deals with the properties of triangles, pencils and circles. It is an abbreviated form of *A Course of Plane Geometry for Advanced Students*, Part II., published in 1910. Naturally the passage of time has suggested re-arrangements, and the helpful criticisms of friends have made it more concise. It is hoped also that the collection of riders has benefited by the omissions and additions which have been made. These still remain much more numerous than any single student will require. If, however, the book is being used with a class, the significance of a particular theorem or group of theorems can frequently be emphasised by taking a sequence of dependent riders rapidly on the blackboard, asking for suggestions, and so working through orally half a dozen or more simple applications in a few minutes. This is a useful variation to the slow progress made in individual work, although it is not a substitute for it; but the method can only be used if there is an ample supply of examples available.

The subject matter has been arranged so as to enable the reader to understand the principles of general projection with real and imaginary elements at as early a stage as possible, not only because it furnishes him with a powerful weapon of attack, but much more because the educational value of higher geometry lies in that novelty of idea and generality of conception which characterise this aspect of the subject, and which are indeed the source of the attraction it has for many students. The variety of application and illustration certainly makes it an entertaining subject to teach, and should save it from being dull to learn.

A companion volume containing answers and solutions has been prepared. Although not a complete key, it is believed that the hints are sufficiently numerous to meet all ordinary requirements: complete solutions are given in the case of the more difficult riders.

A supplementary volume, entitled *A Concise Geometrical Conics*, will be published shortly. It contains a brief account of those properties of the conic which are best treated without the use of projective methods.

The author welcomes this opportunity of acknowledging gratefully the assistance he has received at various times from Mr. R. M. Wright and Mr. R. V. H. Roseveare.

C. V. D.

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PROJECTIVE GEOMETRY

CHAPTER I

ANALYTICAL IDEAS IN PLANE GEOMETRY

THE application of algebraic methods to geometrical research by Descartes (1596–1650) opened up a new line of advance, and thereby gave a wonderful impetus to the mathematicians of his age. It laid bare certain general principles which correlate a large number of apparently dissimilar theorems, and so consolidated in method and enunciation the isolated achievements of independent thinkers. Moreover, it supplied a fruitful basis for further discoveries, by expressing in a concrete analytical form the fundamental laws governing the structure of lines and curves.

In order to enunciate analytical theorems with the utmost generality of which they are capable, it is necessary to enlarge the conceptions in which they are to find expression. The notion of *number*, starting from the instinctive idea of positive integers, has been gradually extended so as to embrace successively fractional, negative, irrational and finally complex numbers, required respectively for the solution of the equations

$$ax=b, \quad ax+b=0, \quad ax^2=b; \quad (ax+b)^2+c=0,$$

where a, b, c represent any positive integers whatever.

With this notion of number, it is possible, but not otherwise, to formulate the general theorem that every equation of degree n has n roots. This instance serves to illustrate the principle that valuable

generalisations are frequently rendered possible by the removal of certain (possibly unconscious) limitations which had been previously imposed. Thus, if complex numbers are regarded as inadmissible, it becomes untrue to say that every equation has a root. Now it may be urged that it is easy to form a clear conception of numbers such as 2, -3, $\frac{2}{3}$, $\sqrt{5}$, but that a number such as $\sqrt{-3}$ is unintelligible. The reason why it is usual to call numbers of the first kind *real* and those of the second kind *imaginary* or *impossible*, is due to the fact that methods of representing graphically numbers of the first type are familiar, but there is less readiness to associate complex numbers with geometrical concepts. Anyone who has drawn a right angled triangle with sides 1 inch, 2 inches, and understands Pythagoras' theorem, feels he has a clear idea of what $\sqrt{5}$ represents. But the Argand diagram and the graphical methods of representing the laws of combination of complex numbers remain outside the elementary syllabus. Such geometrical assistance is of course immaterial; the existence of a number does not depend on the possibility of representing it geometrically, any more than the validity of the proof of a theorem in geometry depends on drawing an accurate or indeed any figure. A figure merely makes it easier to follow the argument; it is a luxury, not a necessity.

Now complex numbers obey the same fundamental laws of Algebra as real numbers, and may therefore enter with equal freedom into analytical processes. If the result of any sequence of such processes is capable of expression in geometrical language, then its enunciation in geometrical terms is valid, even if any part of the process cannot be represented graphically. For since the process is valid analytically, its conclusions must be true.

It is, however, of value to widen the basis of geometrical reasoning by borrowing from analysis notions which are incapable of graphical depiction. This procedure is usually referred to as the **Principle of Continuity.** The germ of this principle dates back to the time of Kepler (1571-1630), who recognised without the aid of analysis that the different species of conics were not isolated curves having each a geometry peculiar to itself but formed a continuous chain, the ellipse passing into the parabola and the parabola into the hyperbola

The first complete exposition was given by Boscovich (1711-1787) in what was intended to form an elementary school text book on conic sections. But it was Poncelet's (1788-1867) introduction of the "circular points at infinity" which revealed in all its stimulating generality the wide range of application of this principle; we shall examine first of all its analytical aspect.

The straight line $x\cos\alpha + y\sin\alpha = p$ cuts the circle $x^2+y^2=a^2$ at the points $(p\cos\alpha \pm \sin\alpha\sqrt{a^2-p^2},\ p\sin\alpha \mp \cos\alpha\sqrt{a^2-p^2})$. These coordinates are real if $p<a$, are coincident if $p=a$, and are complex if $p>a$. By making use of the enlarged conception of *number*, it is possible to enunciate the general theorem that every straight line meets every circle at two "points." This attaches a new significance to the term "point" by removing a limitation, previously imposed, that points must be real, *i.e.* have real coordinates. Any pair of real numbers, *e.g.* $x=2\frac{1}{2}$, $y=-1\frac{1}{3}$, can be represented graphically by the position of a point referred to real coordinate axes, and so can be associated and identified geometrically with a point. If either or both the numbers of a given pair are complex, for the sake of continuity of expression, the pair is still identified with a point called an *imaginary point*, and necessarily incapable of direct graphical representation. The usage of the term is merely conventional, and its importance is simply due to the fact that it corresponds to a definite analytical feature. In the same way the circle $x^2+y^2=a^2$, which is the locus of all "points" at distance a from the origin, does not merely include such points as appear in the figure, but all points which correspond to a pair of numbers, real or complex, satisfying that equation.

Again, the aggregate of pairs of values of x, y which satisfy the equation $ax+by+c=0$, where a, b, c are any constant numbers, constitutes the straight line associated with that equation. A straight line may therefore contain only one point or no points at all, capable of ordinary graphical representation, *e.g.* the line $x-2+\sqrt{-1}(y-3)=0$ contains only one real point (2, 3) and the line $x+y=\sqrt{-1}$ contains no real *finite* point.

By extending the meaning of geometrical terms, it is thus possible to enunciate theorems with greater generality. But in addition to

this, it may enlarge the range of application of a proof, which apparently applies only to a special case For example, if two circles intersect at real points it is easy to show that the locus of a point whose powers w r t the two circles are equal is a straight line, viz the common chord This is expressible in analytical terms, and since the analytical method of proof takes no account of whether the points of intersection are real or imaginary, it follows that the theorem must be true in general In other words, the proof of the general case is implicitly contained in that of the special limited case Free use of imaginary points and lines may therefore be made in order to generalise the range of application of geometrical theorems: and this is referred to under the name of the **Principle of Continuity.** The application of the idea of a limit in geometry is another feature of this principle bearing indeed more directly on Kepler's line of thought, but it would be out of place to develop it here

For the sake of brevity we shall in future employ the following abbreviations

(i) $+\sqrt{-1}$ or $\left(1, \frac{\pi}{2}\right)$ will be denoted by i,

(ii) w r t stands for "with respect to",

(iii) the word "line" will be used for "straight line"

Definitions.

(1) Any pair of numbers, real or complex, is said to be represented by and identified with a *Point* in a given plane, referred to given axes

(2) Any aggregate of pairs of numbers, real or complex, which satisfy the equation $ax+by+c=0$ where a, b, c are any constant numbers, real or complex, is said to constitute a *Straight Line* in a given plane

If any or all the constants a, b, c are complex, and if the relation is altered by writing everywhere $-i$ for i, the straight line is said to be *imaginary* otherwise it is *real* *E g.* $x+2iy=3$ is imaginary, but $(2+3i)x+(10+15i)y=20+30i$ is real

(3) If P is an imaginary point and if a point P is obtained by

writing $-i$ for i in the coordinates of P, then P and P′ are called *conjugate imaginary points*, e g.

$$(x_1 + ix_2, y_1 + iy_2) \quad \text{and} \quad (x_1 - ix_2, y_1 - iy_2),$$

where x_1, x_2, y_1, y_2 are real

(4) If l is an imaginary line, and if a line l' is obtained by writing $-i$ for i in the equation of l, then l and l' are called *conjugate imaginary lines*, e g

$$(a + ia')x + (b + ib')y + c + ic' = 0, \quad (a - ia')x + (b - ib')y + c - ic' = 0,$$

where a, b, c, a', b', c' are real

(5) *A curve whose equation is of the second degree is called a* **Conic** Its most general equation is therefore

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

Note. (i) From the definition of a straight line, it follows that any two straight lines, $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$, meet at one and only one point, except when $\frac{a_1}{b_1} = \frac{a_2}{b_2}$, this exception (the case of parallelism) will be discussed later

(ii) Curves of the second (and higher) degree will be called real if all the coefficients in their equations are real In general we shall deal only with curves which are real in this sense This limitation *does not, of course, secure the existence of real points on the curve*, *e g* $x^2 + y^2 + 1 = 0$ contains no real point

The examples in the following Exercise which are printed in heavy type should be regarded as standard theorems; many of them are best proved by analytical methods

EXERCISE I. a.

1. Prove that two conjugate imaginary lines meet at a real point.

2. Prove that the join of two conjugate imaginary points is a real line.

3. Prove that in general every imaginary line contains one and only one real point What is the limitation ?

4. Prove that one and only one real line can be drawn through a given imaginary point

5. Prove that there are an unlimited number of imaginary points on every given real line and that they occur in conjugate pairs.

6 Prove that there are an unlimited number of imaginary lines passing through every given real point and that they occur in conjugate pairs.

7. If two lines l_1, l_2 meet at a point P and if the conjugate imaginary lines l_1', l_2' meet at P′, prove that P and P′ are conjugate imaginary points.

8. Prove that any real line meets any real conic in two points which are real, coincident or conjugate imaginaries.

9. Prove that two conics intersect at four points which are either all real or two real and two conjugate imaginaries or two pairs of conjugate imaginaries.

10. Find the equation of the real line passing through (i) $3, 4+i$; (ii) $2+3i, 3-2i$.

11. Find the real point on $(2+3i)x+(3-2i)y=8-i$.

12. If the vertices of a quadrangle are two pairs of conjugate imaginary points, prove that the diagonal point triangle is real. [Use Ex. 1·7.]

13. Enunciate and prove the dual of Ex. 12.

14. If two of the vertices of a quadrangle are real points, and if the other two are conjugate imaginaries, prove that one vertex and the opposite side only of the diagonal point triangle are real. What is the dual theorem ?

15. A quadrilateral is formed by drawing the tangents from two conjugate imaginary points to a given real conic, and the diagonals are drawn : determine which points and lines in the figure are real and which are conjugate imaginaries.

16. What do the following equations represent ?

(i) $x^2+y^2=0$. (ii) $x^2+y^2-6x-8y+21=0$.

(iii) $x^2+y^2-6x-8y+25=0$. (iv) $x^2+y^2-6x-8y+29=0$.

17. Prove that in general an imaginary conic contains either 0 or 2 or 4 real points, but not more than four real points. Discuss the case illustrated by $2x^2+xy(i-2)-iy^2-x+y=0$.

18. If a conic whose equation is real passes through one real point, prove that it passes through an unlimited number of real points.

Angle between two Lines. In the case of real lines we know that $\dfrac{m_1-m}{1+m_1m}$ represents the tangent of the angle which $y=m_1x+c_1$ makes with $y=mx+c$. If m is given and if the value of this expression is given, there is in general one and only one value of m_1.

Thus, if $\dfrac{m_1-m}{1+m_1m}=k$, we have

$$m_1-m=k(1+m_1m) \quad \text{or} \quad m_1(1-km)-(m+k)=0.$$

This gives a unique value for m_1, unless $1-km=0$ *and* $m+k=0$, in which case $1+m^2=0$ or $m^2=-1$ or $m=\pm i$. If, however, $m=\pm i$ we have $k=\mp i$, and any value whatever of m_1 satisfies the relation

Definition. If either or both the lines $y=mx+c$, $y=m_1x+c_1$ are imaginary, the expression $\dfrac{m_1-m}{1+m_1m}$ is defined to be the *tangent of the angle* which $y=m_1x+c_1$ makes with $y=mx+c$, *provided that* neither m^2 nor m_1^2 is equal to -1.

Starting from this definition, there is no difficulty in building up the trigonometrical formulae for imaginary angles or angles between imaginary lines, provided that no attempt is made to associate the lines $y=\pm ix$ with the idea of direction. On account of this property, Laguerre has called the lines $y=\pm ix+c$ **isotropic.**

Cross ratio of four Lines. If four real concurrent lines, referred to their common vertex as origin, are represented by $y=m_1x$, $y=m_2x$, $y=m_3x$, $y=m_4x$, it is easy to prove that their cross ratio is equal to $\dfrac{(m_2-m_1)(m_4-m_3)}{(m_4-m_1)(m_2-m_3)}$, if this expression equals -1, the four lines form a harmonic pencil and the line pair $y=m_1x$, $y=m_3x$ is harmonically conjugate to the line pair $y=m_2x$, $y=m_4x$.

If any or all of these lines are imaginary, we *define* their cross ratio as $\dfrac{(m_2-m_1)(m_4-m_3)}{(m_4-m_1)(m_2-m_3)}$ and *define* a pencil as harmonic if its cross ratio equals -1.

Laguerre's Theorem. *If a is the angle between the lines $y=mx$, $y=m'x$, then the cross ratio of the pencil $y=mx$, $y=ix$, $y=m'x$, $y=-ix$ equals $\cos 2a+i\sin 2a$.*

Let $m=\tan\theta$, $m'=\tan\theta'$, so that $a=\theta-\theta'$.

$$\text{The cross ratio}=\frac{(i-m)(-i-m')}{(-i-m)(i-m')}=\frac{(1+im)(1-im')}{(1-im)(1+im')}$$

$$=\frac{(\cos\theta+i\sin\theta)(\cos\theta'-i\sin\theta')}{(\cos\theta-i\sin\theta)(\cos\theta'+i\sin\theta')}=\frac{\cos(\theta-\theta')+i\sin(\theta-\theta')}{\cos(\theta-\theta')-i\sin(\theta-\theta')}$$

$$=\cos 2a+i\sin 2a.$$

In particular, if $a=\dfrac{\pi}{2}$, the cross ratio $=-1$, and if

$$\cos 2a+i\sin 2a=-1,$$

we have $\quad \cos 2a=-1,\ \sin 2a=0, \quad$ and so $a=\dfrac{\pi}{2}$.

We may state these results geometrically as follows

Theorem 1. (i) Any pair of perpendicular lines is harmonically conjugate to the isotropic lines.

(ii) If a variable pair of lines contain a constant angle, they form with the isotropic lines a pencil of constant cross ratio

(iii) If two lines are harmonically conjugate to the isotropic lines, they must be perpendicular to each other

(iv) If a variable pair of straight lines form with the isotropic lines a pencil of constant cross ratio, they must make a constant angle with each other

EXERCISE I. b.

1. Evaluate $\dfrac{m_1 - m}{1 + mm_1}$ if $m_1 = i$

2 If $\angle ABC = 90^\circ$, $AB = i$, $BC = 1$, what is AC ?

3 If $\tan\theta = i$, what values would the usual formulae assign to $\sin\theta$ and $\cos\theta$?

4. What does the equation of the line $y = ix$ become when the axes (rectangular) are rotated through an angle α ?

5 Interpret $x^2 + y^2 = 0$ in two different ways

6 The coordinates of A, B, C are respectively $(3, 3i)$, $(0, 0)$, $(2, 2i)$, what lengths would the usual formulae attribute to AB, BC CA ? Which is the greater, $(AB + BC)^2$ or AC^2 ?

7 Work out the cross ratio of the lines

$$y = mx, \quad y = ix \quad y = -\frac{1}{m}x, \quad y = -ix$$

8. Show that $ax^2 + 2hxy - ay^2 = 0$ is harmonically conjugate to $x^2 + y^2 = 0$

9 **Prove that the isotropic lines are the only lines which are harmonically conjugate to each of two pairs of perpendicular lines** [What is the condition that $ax^2 + 2hxy + by^2 = 0$ is harmonically conjugate to

$$x^2 + 2pxy - y^2 = 0 \quad \text{and to} \quad x^2 + 2qxy - y^2 = 0 \text{ ?}]$$

Transformation. It is often possible to deduce from one known property another property, apparently of an entirely different character, by effecting a correspondence between two distinct geometrical systems Two examples of this process of transformation are no doubt familiar to the reader (i) Inversion, (ii) The

Principle of Duality. Each of these processes depends on a definite law of correspondence which is used to transform a given geometrical system into a new system, and thereby establishes in the new system the existence of properties which correspond to known properties of the original system.

Another and more important example is supplied by the process known as **conical projection.** Consider any figure C, consisting of points, lines or curves situated in a plane Σ. Lines are drawn from a given point O outside Σ to all these points and to all the points on the lines or curves of C. These lines, radiating from O, are cut by any other plane Σ' in a new system of points, lines or curves forming a figure C′. Then C′ is said to be the conical projection of C, and O is called the **vertex of projection.** Now suppose that both systems C, C′ are placed in the same plane. We then have two coplanar systems so related that to *every* point and *every* straight line of C corresponds one and only one point and one and only one straight line of C′ *and vice versa*. We then say that there exists a one to one or (1, 1) correspondence between C and C′. A transformation which when applied to a plane system C generates a plane system C′ such that C and C′ are in (1, 1) correspondence is called a **homographic transformation.**

The general analytical expression for a homographic transformation is of great importance.

Let (x, y) be the coordinates of a point A in the plane system C, and let (ξ, η) be the coordinates of the corresponding point A′ in the coplanar system C′, where C and C′ are in (1, 1) correspondence.

It is required to express ξ and η in terms of x, y. Since the correspondence is (1, 1), the required expressions cannot contain any radicals, and we shall assume that they do not involve transcendental functions. Reducing them to a common denominator, we write

$$\xi = \frac{f_1(x, y)}{\phi(x, y)}, \quad \eta = \frac{f_2(x, y)}{\phi(x, y)},$$

where f_1, f_2, ϕ are polynomials in x, y such that there is no factor common to all three functions f_1, f_2, ϕ.

Then to any straight line $p\xi + q\eta + r = 0$ corresponds

$$pf_1(x, y) + qf_2(x, y) + r\phi(x, y) = 0.$$

But by hypothesis this is a straight line Therefore f_1, f_2, ϕ must be linear in x, y,

$$\xi = \frac{a_1 x + b_1 y + c_1}{lx + my + n}, \quad \eta = \frac{a_2 x + b_2 y + c_2}{lx + my + n}.$$

If we solve these equations for x and y, we see that the solutions are of the form

$$x = \frac{\alpha_1 \xi + \beta_1 \eta + \gamma_1}{\lambda \xi + \mu \eta + \nu}, \quad y = \frac{\alpha_2 \xi + \beta_2 \eta + \gamma_2}{\lambda \xi + \mu \eta + \nu}.$$

And it is obvious that these relations are in general sufficient to secure the required (1, 1) correspondence Points on the lines $lx + my + n = 0$ and $\lambda \xi + \mu \eta + \nu = 0$ require further consideration But leaving aside these important special cases, we may say that the relations obtained above provide the most general analytical expressions for a homographic transformation.

If a point A in C coincides with the corresponding point A in C it is called a *self-corresponding point* of the two systems

Theorem 2. The cross ratio of any pencil of four concurrent lines is unaltered in value by any homographic transformation.

Take the origin at the point of concurrency so that the four lines of the ξ, η system may be written

$$\eta = m_1 \xi, \quad \eta = m_2 \xi, \quad \eta = m_3 \xi, \quad \eta = m_4 \xi$$

With the notation used above, the line corresponding to $\eta = m_1 \xi$ is

$$a_2 x + b_2 y + c_2 = m_1 (a_1 x + b_1 y + c_1),$$

which is parallel to $y = -\dfrac{a_2 - m_1 a_1}{b_2 - m_1 b_1} x = \mathsf{M}_1 x$, say

Let the other corresponding lines be parallel to

$$y = \mathsf{M}_2 x, \quad y = \mathsf{M}_3 x, \quad y = \mathsf{M}_4 x$$

Then
$$\mathsf{M}_2 - \mathsf{M}_1 = \frac{a_2 - m_1 a_1}{b_2 - m_1 b_1} - \frac{a_2 - m_2 a_1}{b_2 - m_2 b_1}$$
$$= \frac{(m_2 - m_1)(a_1 b_2 - a_2 b_1)}{(b_2 - m_1 b_1)(b_2 - m_2 b_1)},$$

$\dfrac{(\mathsf{M}_2 - \mathsf{M}_1)(\mathsf{M}_4 - \mathsf{M}_3)}{(\mathsf{M}_4 - \mathsf{M}_1)(\mathsf{M}_2 - \mathsf{M}_3)}$ reduces at once to $\dfrac{(m_2 - m_1)(m_4 - m_3)}{(m_4 - m_1)(m_2 - m_3)}$

Corollary 1. The cross-ratio of any range of four collinear points is unaltered in value by any homographic transformation.

For the cross-ratio of a range is equal to that of any pencil having this range as a section.

Corollary 2. If a homographic transformation is applied to a harmonic pencil or to a harmonic range the new pencil or the new range is harmonic.

Theorem 3. Any conic may be transformed homographically into a circle by a relation of the form

$$x=\frac{p\xi}{\eta+p}, \quad y=\frac{q\eta}{\eta+p}.$$

Let the conic be $ax^2+2hxy+by^2+2gx+2fy+c=0$.

The transformed curve is

$$ap^2\xi^2+2hpq\xi\eta+bq^2\eta^2+2gp\xi(\eta+p)+2fq\eta(\eta+p)+c(\eta+p)^2=0.$$

This is a circle if

$$ap^2=bq^2+2fq+c \quad \text{and} \quad 2hpq+2gp=0,$$

which hold good if

$$q=-\frac{g}{h} \quad \text{and} \quad p^2=\frac{1}{ah^2}(bg^2-2fgh+ch^2).$$

The transformation can therefore be effected in this way: it is, of course, clear that there are far more general forms of transformation which also produce the required result.

EXERCISE I. c.

1. Prove that the degree of a curve is unaltered by any homographic transformation.

2. How many independent constants are there in the general homographic transformation? Show that a transformation can be chosen which will make any given quadrangle correspond to any other given quadrangle. What happens if the four corners of the quadrangle degenerate into four collinear points?

3. Find the self-corresponding points in the transformation defined by

$$\xi=\frac{x+2y}{x+y-1}, \quad \eta=\frac{2x+y}{x+y-1}.$$

4. If two coplanar figures correspond homographically, prove that there are in general three (finite) self-corresponding points. How many self-corresponding lines exist?

5. If the relation is $\xi=\frac{x+3y-1}{x+2y+1}$, $\eta=\frac{2x-y+1}{x+2y+1}$, prove that all lines in the (ξ, η) system parallel to $\xi+2\eta=0$ correspond to lines in the (x, y) system which concur at $(-\frac{1}{9}, -\frac{4}{9})$

6. With the notation used above, prove that a set of parallel lines in the (ξ, η) system correspond to a set of lines in the (x, y) system which concur at a point on $lx+my+n=0$

7. Prove that the homographic relation $\xi=\dfrac{p-\frac{q^2}{a}x}{x}$, $\eta=\dfrac{qy}{x}$ transforms the parabola $\eta^2=a\xi$ into a circle

8 If the homographic relation is $\xi=a_1x+b_1y+c_1$ $\eta=a_2x+b_2y+c_2$, prove that parallel lines correspond to parallel lines

9. What simple homographic relations will make

$$\text{(i)}\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \text{and} \quad \text{(ii)}\ \frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

correspond to the circle $\xi^2+\eta^2=c^2$?

The Line at Infinity. The word ' infinity ' occurs so often in analytical and geometrical reasoning that a definition of the concept it represents is essential To say that infinity is a number larger than any number we can think of is nonsense It is a manifest contradiction to talk of imagining a number larger than any number that can be thought of This then, at least is true—**infinity is not a number.** Consider a series of points A_0, A_1, A_2, … A … marked off at unit distance apart, along an endless straight line If we proceed to count the number of points marked off on the line, it is clear that it is impossible ever to complete the process , and accordingly it is customary to say that the number of points is unlimited or infinite The word " infinite," then, does not imply a numerical measure of the points on the line, but describes the *nature* of the counting process It is easy to formulate a test which distinguishes between a finite and an infinite group Let us suppose we are counting the members of a group, one by one , if the group is finite it is possible for an opponent to name a definite number N such that the counting process will be completed before the number N is attained. But if the group is infinite, no matter what definite number N our opponent cares to select, the counting process will eventually yield numbers greater than N

Consider now the homographic transformation

$$\xi = \frac{x}{1-x}, \quad \eta = \frac{y}{1-x}$$

Suppose a point P starting at the origin in the (x, y) system moves along the x axis towards the points $(1, 0)$ then in the (ξ, η) system the corresponding point P also starts from the origin and moves along the ξ axis Now the more closely x approaches the value 1, the larger ξ becomes: whatever number N our opponent chooses however large, it is possible to find a value of x near 1 such that $\xi > \mathrm{N}$ for this value of x and for all other values of x between this value and 1 Under these circumstances we say that ξ tends to infinity when x tends to 1 from below, and using Dr Leathem's notation we write $\xi \to \infty$ when $x \nearrow 1$ If $x > 1$, ξ is negative, whatever negative number $-\mathrm{N}$ our opponent chooses, however large, it is possible to find a value of x near 1 such that $\xi < -\mathrm{N}$ for this value of x and for all other values of x between this value and 1 We therefore say that ξ tends to minus infinity when x tends to 1 from above and write $\xi \to -\infty$ when $x \searrow 1$ The symbol ∞, an endless loop, is due to John Wallis (1616 1703), Savilian Professor of Geometry at Oxford and the inventor of negative indices These statements constitute the definition of the sense in which the word "infinity" is to be used they show that (i) ∞ is not a number and (ii) it is nonsense to say that anything can *equal* infinity

We shall now examine the homographic transformation

$$\xi = \frac{x}{1-x}, \quad \eta = \frac{y}{1-x}$$

in greater detail, we see that to every point in the x, y system there corresponds one and only one point in the ξ η system, except for points on the line $x=1$ But for points on the line $x=1$, ξ ceases to have any corresponding numerical value The correspondence between the two systems is marred by this discontinuity, that there exists a single line $x=1$ in the x y system to which there is no corresponding line in the (ξ, η) system To secure generality of statement, this exceptional case must be eradicated We therefore invent a line which will be called the "*ideal line*" or the "*line at infinity*" and add this to the (ξ, η) system This line is essentially

fictitious. It is created solely for the purpose of corresponding to the line $x=1$, and possesses only such properties as are consequent from this correspondence The name "line at infinity" is indeed misleading, for (i) it is not a line at all in the ordinary sense, no idea of direction can be associated with it, and (ii) infinity is not a geographical description as the phrase suggests.

Further, we conceive a series of "*ideal points*" or "points at infinity" as composing the line at infinity, their existence being justified solely by the fact that they fulfil the function of corresponding to points on the line $x=1$

If we now solve for x, y in terms of ξ, η, we see that

$$x=\frac{\xi}{\xi+1}, \quad y=\frac{\eta}{\xi+1}$$

Exactly as before it appears that to every point in the ξ, η system there corresponds one and only one point in the x, y system, except for points on the line $\xi+1=0$ Hence in order to complete the correspondence, it is necessary to add to the x, y system a fictitious line which, as before, is called the "ideal line" or the "line at infinity" in the x, y system. By this final addition, the (1, 1) correspondence between the two systems is made complete, without exception

Suppose we have a set of concurrent lines in the x y system which concur at a point A on the line $x=1$, say at $x=1$ $y=b$ Then they correspond to a set of lines in the ξ, η system which have no finite point of intersection, *i e* to a set of parallel lines Analytically the lines in the x, y system are $\lambda(x-1)+(y-b)=0$ where λ varies These become

$$\lambda\left(\frac{\xi}{\xi+1}-1\right)+\left(\frac{\eta}{\xi+1}-b\right)=0$$

or

$$-\lambda+\eta-b(\xi+1)=0,$$

which represents a set of parallel lines when λ varies Also the point of concurrency A in the x, y system corresponds to an ideal point a in the ξ, η system We therefore regard the set of parallel lines in the ξ, η system as containing and determining the ideal point a Further, no line contains more than one ideal point, because the line corresponding to it in the x, y system meets the line $x=1$ at one and

only one point. We therefore associate the direction of a line or the direction of a set of parallel lines with the ideal point so determined. By a convenient extension of the meaning of words, we say that parallel lines cut the line at infinity at the same (ideal) point. It should be noted that the statement that a line contains one and only one ideal point is the equivalent of Playfair's Axiom, and we have deduced it from analytical considerations, which tacitly assume the Euclidean axiom of parallelism. If, as in non-Euclidean geometry, this axiom is discarded it is no longer true to say that every line contains *either* one *or* only one ideal point. Summarising this discussion, we conclude by saying that the term "line at infinity" is introduced solely to secure continuity of statement in dealing with homographic transformations. It is a straight line, only in the sense that it corresponds homographically to a straight line; and this involves the fact that it meets every other straight line in one and only one (ideal) point. Further, it lies at infinity, only in the sense that it contains no finite point; the statement that it lies at infinity describes its character, not its geographical position; and lastly, it is devoid of any sense of direction; being the aggregate of all ideal points, it is the aggregate of all possible directions.

Analytical Representation of Points at Infinity.

Let the coordinates of any point P referred to rectangular axes

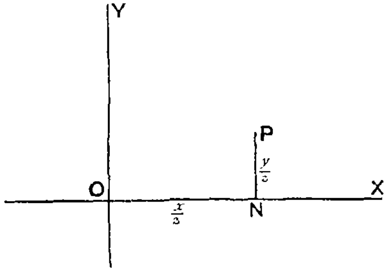

FIG. 1

OX, OY be $\left(\frac{x}{z}, \frac{y}{z}\right)$, where x, y, z are always connected by the relation

$$x+y+z=1.$$

To every position of P there corresponds a unique set of values for x, y, z. And any straight line can be represented by an equation of the form

$$a\left(\frac{x}{z}\right)+b\left(\frac{y}{z}\right)+c=0$$

or

$$ax+by+cz=0.$$

Any two parallel lines can be represented by

$$y-mx=c_1z, \quad y-mx=c_2z.$$

To obtain their (ideal) point of intersection we have by subtraction $(c_1-c_2)z=0$ or $z=0$.

Hence all ideal points satisfy the condition $z=0$: we therefore say that $z=0$ is the *equation of the line at infinity* in the plane considered. We may also write the coordinates of the ideal point determined by these parallel lines in the form $z=0$, $y-mx=0$, or

$$\frac{x}{1}=\frac{y}{m}=\frac{z}{0}=\frac{1}{1+m},$$

since $x+y+z=1$; the coordinates are therefore

$$\frac{1}{1+m}, \quad \frac{m}{1+m}, \quad 0$$

Again, the equation of any circle can be written in the form

$$x^2+y^2+2gxz+2fyz+cz^2=0.$$

This meets $z=0$, where $x^2+y^2=0$ or $y=\pm ix$. Hence *every circle cuts the line at infinity in the same two points*, viz.

$$\frac{x}{1}=\frac{y}{\pm i}=\frac{z}{0}=\frac{1}{1\pm i}$$

These two points are called the **circular points at infinity**, and wil be denoted by ω, ω'.

Theorem 4. Two concentric circles have double contact wit each other at the circular points at infinity.

Take the origin at their common centre. Their equations tak the form $x^2+y^2-a^2z^2=0$, $x^2+y^2-b^2z^2=0$. Therefore their inte sections are given by $x^2+y^2=0$, $z^2=0$

$\therefore$ they touch each other at the points $\frac{x}{1}=\frac{y}{\pm i}=\frac{z}{0}$, *i e.* at ω, ω'.

Theorem 5. If two conics are homographically related to two concentric circles, the two conics must have double contact with each other

A contact can be regarded as the limit of an intersection at two adjacent points, hence if two curves touch each other, any two curves homographically related to them must also touch each other. But two concentric circles touch each other at ω, ω, therefore the conics touch each other at the points P, P′, which correspond to ω, ω'.

Theorem 6. Every conic which passes through the circular points at infinity must be a circle

The equation of any conic can be written in the form

$$ax^2+2hxy+by^2+2gxz+2fyz+cz^2=0$$

This meets $z=0$, where $ax^2+2hxy+by^2=0$

But it is given that it meets $z=0$ where $x^2+y^2=0$,

$a=b$ and $h=0$, the conic is a circle

Although the last few theorems have been stated in geometrical terms, it must be clearly understood that such geometrical language is employed merely as a convenient descriptive means of expressing the result of an analytical process. And it is only possible to make use of this geometrical notation if it is agreed initially that the terms involved have a wider significance than is attached to them in ordinary graphical work The existence of the circular points at infinity does not correspond to any graphical reality, but merely affords a conventional and suggestive interpretation by a geometrical channel of an analytical phenomenon It expresses a definite feature of the properties of a circle, defined analytically for rectangular axes, viz the terms of the second degree in the equation of every circle are of the form $a(x^2+y^2)$ and conversely if an equation is of the second degree with $x^2+y^2+0\ xy$ as its leading terms it must represent a circle But while it is convenient to generalise our notation, it is at the same time necessary to avoid making other uses of this notation than are warranted by the underlying analytical structure We speak of the circular points at infinity as "points" because they can be made to correspond homographically to two finite points (see Exercise I d, No 4), and this form of words is useful because certain properties can be transmitted from one system to

another by a homographic transformation. We shall obtain later the analytical form of the homographic transformation corresponding to a conical projection. Since the constants which appear in the equations of transformation may be real or complex, the foregoing pages supply a complete justification for the validity of imaginary projection, *i.e.* projection where either the vertex or any points or lines in the projected figure are imaginary. The process is, in fact, to be regarded as analytical in its nature. The statement of the mode of projection in geometrical language is only a convenient means of indicating the way in which the homographic relation is to be chosen

EXERCISE I. d.

1. For the transformation $\xi = \frac{2x - y + 1}{x + y}$, $\eta = \frac{x + 2y - 1}{x + y}$, find the line in the x, y system and the line in the ξ, η system which corresponds to the line at infinity in the other

2. Find the general homographic transformation such that the lines corresponding to the line at infinity are $x = 0$ and $y = 0$

3. Find the conic which corresponds to the circle $x^2 + y^2 = 1$ for the transformation $x = \frac{2\xi + \eta}{\xi - 1}$, $y = \frac{\xi - 2\eta}{\xi - 1}$

4. For the transformation $\xi = \frac{bx}{y - b}$, $\eta = \frac{ify}{y - b}$, show that the circular points at infinity in the ξ, η system correspond to the points (f, b), $(-f, b)$ in the x, y system

5. Find a simple transformation which will make the points $(a, 2a)$, $(a, -2a)$ correspond to the circular points at infinity, and verify that it converts the parabola $y^2 = 4ax$ into a circle

6. What are the x, y, z coordinates of (i) the point $(3, 4)$, (ii) the point at infinity on $2x + 3y = 1$, (iii) the points at infinity on $4x^2 - y^2 = 1$?

7. $S_1 = 0$, $S_2 = 0$ are the equations of two conics; prove that any homographic transformation which makes them both correspond to circles will also make the conic $S_1 - \lambda S_2 = 0$ correspond to a circle, where λ is any constant State this result in geometrical terms

8 Determine a simple homographic relation which will transform both the conics $x^2 + 3y^2 = 1$ and $3x^2 + y^2 = 2$ into circles

9. Prove that the transformation caused by 'inversion' may be put in the form $\xi = \frac{k^2 x}{x^2 + y^2}$, $\eta = \frac{k^2 y}{x^2 + y^2}$, where k is the radius of inversion. Is this a $(1, 1)$ correspondence?

CHAPTER II

ORTHOGONAL PROJECTION

IMAGINE any geometrical system of points, lines and curves traced on a plane glass plate If this is held up in the sunlight above a sheet of paper, shadows will be cast on the paper forming a second geometrical system corresponding to the first If the paper is held so that all the light rays are perpendicular to the plane of the paper, the shadow system is called the **orthogonal projection** of the system on the glass plate There is a (1, 1) correspondence between the two systems, but they differ in shape and size, if the plane of the glass is not parallel to the plane of the paper.

Definitions.

(i) If P_1, P_2, are a system of points in a plane Σ, and if p_1, p_2, are the feet of the perpendiculars from these points to a plane σ, then the system p_1, p_2, is called the **orthogonal projection** of the system P_1, P_2, on the plane σ

(ii) The line of intersection of the planes Σ and σ is called the **axis of projection.**

Some of the following theorems still hold good in the more general process of conical projection, these are indicated by an asterisk In the present chapter, the word "projection" is used as an abbreviation for "orthogonal projection"

Unless otherwise stated, capital letters refer to the original system and small letters denote the corresponding elements of the projected system.

* **Theorem 7.** (1) A straight line projects into a straight line.

(2) The meet of two straight lines or curves projects into the meet of their projections.

(3) The join of two points projects into the join of their projections.

(4) Any point on the axis of projection is unaltered in position by projection.

(5) Any line and its projection meet on the axis of projection.

The proof of these statements is obvious.

Theorem 8. Parallel lines project into parallel lines.

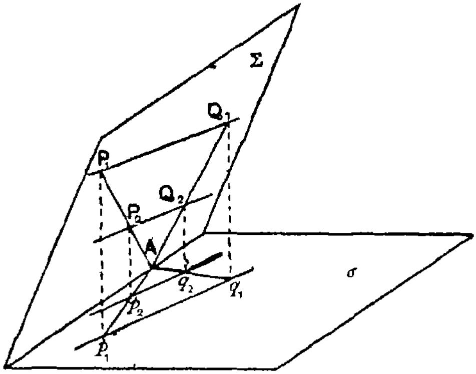

FIG. 2.

Let the parallel lines be P_1Q_1 and P_2Q_2; take any point A on the axis of projection and draw any two lines AP_2P_1, AQ_2Q_1 to cut the parallel lines at P_2, P_1 and Q_2, Q_1. Let p_2, p_1, q_2, q_1 denote their projections on σ.

By parallels, $$\frac{Ap_1}{Ap_2}=\frac{AP_1}{AP_2}=\frac{AQ_1}{AQ_2}=\frac{Aq_1}{Aq_2};$$

$\therefore$ p_1q_1 is parallel to p_2q_2. Q.E.D.

Corollary. The line at infinity in Σ projects into the line at infinity in σ.

For any point at infinity in Σ is determined by a set of parallel

lines in Σ; these project into a set of parallel lines in σ which determine a point at infinity in σ.

The reader should note that this corollary cannot be proved by an argument based on a statement such as the line at infinity is a line at an infinite distance, for this phrase is meaningless. If logical use is to be made of the notion of "infinity," it is essential that properties associated with it should be deduced rigorously from the initial definitions and axioms

Theorem 9. (1) If P_2 is the mid-point of P_1P_3, then p_2 is the mid-point of p_1p_3.

(2) If P_1, P_2, P_3 are collinear, then $\frac{P_1P_2}{P_2P_3}=\frac{p_1p_2}{p_2p_3}$.

*(3) If P_1, P_2, P_3, P_4 are collinear, and if $\{P_1P_2P_3P_4\}$ is harmonic, so is $\{p_1p_2p_3p_4\}$.

*(4) If P_1, P_2, P_3, P_4 are collinear, the ranges $\{P_1P_2P_3P_4\}$ and $\{p_1p_2p_3p_4\}$ are equicross.

*(5) The cross ratio of a pencil is equal to that of its projection.

The proof of these statements is obvious.

Theorem 10. If P_1Q_1, P_2Q_2 are parallel, then $\frac{P_1Q_1}{P_2Q_2}=\frac{p_1q_1}{p_2q_2}$.

Let P_1P_2 meet Q_1Q_2 at A, and let a be the projection of A (cf Fig. 2).

By parallels, $$\frac{P_1Q_1}{P_2Q_2}=\frac{P_1A}{P_2A}=\frac{p_1a}{p_2a}=\frac{p_1q_1}{p_2q_2}.$$ Q.E.D.

Theorems 9 (2) and 10 amount to the statement that the ratios of segments of the same or parallel lines are unaltered by projection.

Theorem 11. (1) If P_1P_2 is parallel to the axis of projection, $P_1P_2=p_1p_2$.

(2) If P_1P_2 is perpendicular to the axis of projection, and if θ is the angle of intersection of the two planes, $p_1p_2=P_1P_2 \,.\, \cos\theta$.

The proof of these statements is obvious.

Analytical Treatment.

Choose the axis of projection as the Y axis in Σ and the y-axis in σ. From any point O as origin on the axis of projection draw OX, Ox perpendicular to the axis of projection in the planes Σ, σ.

Let (X, Y) be the coordinates of any point P in Σ, and let (x, y) be the coordinates of its projection p on σ. Let θ be the angle of inclination of Σ and σ.

Then, by Theorem 11, $x = \mathrm{X} \cos\theta$, $y = \mathrm{Y}$.

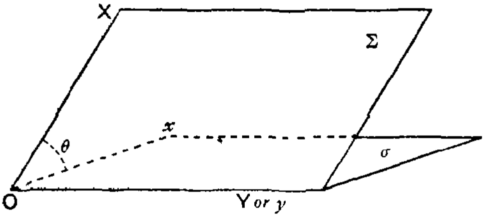

FIG. 3.

These relations constitute a homographic transformation of the simplest possible type.

Theorem 12. If S is the area of any closed figure in Σ, and if s is the area of its projection on σ, then $s = \mathrm{S} \,.\, \cos\theta$.

If any line parallel to OX cuts the boundary of S in P_1, P_2 and OY in N, and if $\mathrm{NP}_1 = \mathrm{X}_1$, $\mathrm{NP}_2 = \mathrm{X}_2$, and if the extreme positions of N are A, B where $\mathrm{OA} = a$, $\mathrm{OB} = b$, we have

$$\mathrm{S} = \int_a^b (\mathrm{X}_1 - \mathrm{X}_2)\,d\mathrm{Y}.$$

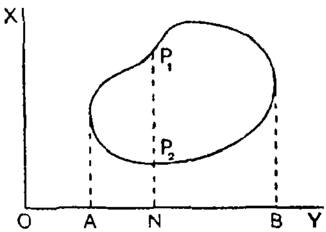

FIG 4

But $\mathrm{X}_1 \cos\theta = x_1$, $\mathrm{X}_2 \cos\theta = x_2$, and $\mathrm{Y} = y$;

$$\therefore\ \mathrm{S} = \int_a^b \frac{x_1 - x_2}{\cos\theta}\,dy = \frac{1}{\cos\theta}\int_a^b (x_1 - x_2)\,dy = \frac{s}{\cos\theta},$$

$$\therefore\ s = \mathrm{S} \,.\, \cos\theta. \qquad \text{Q.E.D.}$$

Theorem 13. The projection of the centroid of an area bounded by a curve C is the centroid of the area bounded by the curve c, the projection of C.

With the same notation as in Theorem 12,

$$x\text{-coordinate of centroid of } c = \frac{\iint x\,dx\,dy}{\iint dx\,dy}.$$

But $x = \mathsf{X}\cos\theta$ and $y = \mathsf{Y}$,

$$\therefore\ x\text{-coordinate of centroid of } c = \frac{\iint \mathsf{X}\cos^2\theta\,d\mathsf{X}\,d\mathsf{Y}}{\iint \cos\theta\,d\mathsf{X}\,d\mathsf{Y}}$$

$$= \cos\theta \, . \, \frac{\iint \mathsf{X}\,d\mathsf{X}\,d\mathsf{Y}}{\iint d\mathsf{X}\,d\mathsf{Y}}$$

$$= \cos\theta \, . \, \mathsf{X}\text{ coordinate of centroid of C}.$$

$$\text{Similarly, } y \text{ coordinate of centroid of } c = \frac{\iint y\,dx\,dy}{\iint dx\,dy}$$

$$= \frac{\iint \mathsf{Y}\cos\theta\,d\mathsf{X}\,d\mathsf{Y}}{\iint \cos\theta\,d\mathsf{X}\,d\mathsf{Y}} = \frac{\iint \mathsf{Y}\,d\mathsf{X}\,d\mathsf{Y}}{\iint d\mathsf{X}\,d\mathsf{Y}}$$

$$= \mathsf{Y}\text{-coordinate of centroid of C},$$

$\therefore$ the centroid of c is the projection of the centroid of C

Q.E.D

The Ellipse. Any equation of the second degree represents a conic (or a pair of straight lines) If the equation is of the form

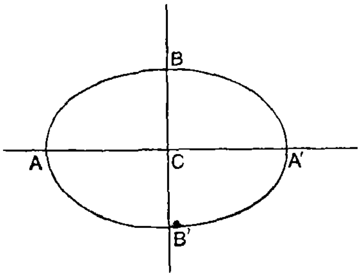

FIG 5

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, it is evident that the curve is symmetrical about each

coordinate-axis and also about the origin. The graph is shown in Fig. 5, where C is the origin, ACA′ the x-axis and BCB′ the y-axis.

If $a > b$, the line ACA′ is called the *major axis* and the line BCB′ is called the *minor axis*, and C is called the *centre* of the ellipse; $AC = CA' = a$, $BC = CB' = b$.

Theorem 14. The ellipse $\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$, where $a > b$, is projected orthogonally into the circle $x^2 + y^2 = b^2$, if the axis of projection is

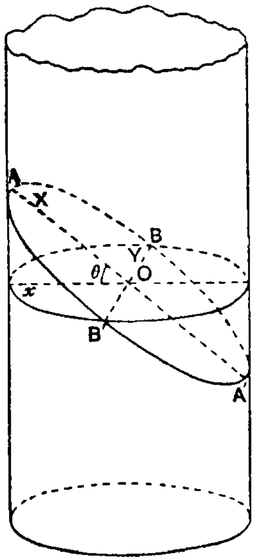

Fig. 6.

the minor axis OY and if the angle θ between the planes is given by $\cos\theta = \frac{b}{a}$.

With the previous notation, $x = X\cos\theta = \frac{bX}{a}$, $y = Y$;

$$\therefore \frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1 \quad \text{becomes} \quad \left(\frac{ax}{b}\right)^2 \cdot \frac{1}{a^2} + \frac{y^2}{b^2} = 1$$

or $$x^2 + y^2 = b^2. \qquad \text{Q.E.D.}$$

Corollary 1. Ellipses, which have their major axes parallel and the ratio of their major to their minor axis constant, can be projected simultaneously into a system of circles.

The equation of any ellipse of the system can be written

$$\frac{(X-p)^2}{\lambda^2 a^2}+\frac{(Y-q)^2}{\lambda^2 b^2}=1,$$

where a, b are constant and λ, p, q vary; the transformation $x=\frac{bX}{a}$, $y=Y$ converts each ellipse into a circle.

Corollary 2. If, in addition, the ellipses are concentric, they can be projected into a system of concentric circles.

Definition. Ellipses which have their major axes parallel and the ratio of their major to their minor axis constant are said to be **homothetic.**

The reader who is acquainted with the geometry of the ellipse will observe that the connection between a circle and its orthogonal projection is identical with the connection between the auxiliary circle of an ellipse and the ellipse itself

It should be noted that if the circle $X^2+Y^2=a^2$ is projected into the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the projection of the point $(a\cos\theta, a\sin\theta)$ on the circle is the point $(a\cos\theta, b\sin\theta)$ on the ellipse, this angle θ is called the *eccentric angle* of the point, and we therefore see that the eccentric angle is unaltered by projection.

Since a tangent to a curve may be regarded as the limiting position of a line cutting the curve, it follows that the projection of a tangent to a curve is a tangent to the projected curve.

EXERCISE II. a.

1. A line AB of length r in the plane Σ makes an angle α with the axis of projection, the angle between Σ and σ is θ, find the length of ab.

2. With the data in Ex 1, prove that ab makes an angle

$$\tan^{-1}(\tan\alpha \quad \cos\theta)$$

with the axis of projection

3. ABC is an equilateral triangle in the plane Σ and BC is parallel to the axis of projection, the angle between Σ and σ is 45°, calculate $\angle abc$

4. Prove from first principles that the projection of the centroid of the triangle ABC is the centroid of abc

5. If the axis of projection is the x axis, and if the angle between the planes is 45°, find the projection of $2x^2 + y^2 = r^2$

6. AB, CD are two equal perpendicular lines of constant length in Σ; prove that $ab^2 + cd^2$ is constant for a given plane of projection

7. ABC is an equilateral triangle of given size, prove that

$$ab^2 + bc^2 + ca^2$$

is constant.

8. Find the angle between the planes if the projection of $y^2 = 4ax$ is $y^2 = 4bx$, (i) if $a < b$, (ii) if $a > b$

9. If, with the notation of Fig 5, N is the foot of the perpendicular from a variable point P on the ellipse to the major axis AA′, and if NP is produced to Q so that $\frac{NP}{NQ} = \frac{b}{a}$, prove that the locus of Q is a circle having AA′ as diameter [This circle is called the *auxiliary* circle]

10 A variable chord of a given ellipse cuts off a segment of constant area, prove that the chord touches a fixed homothetic ellipse

11 If PQR is a triangle of maximum area that can be inscribed in a given ellipse, prove that the tangent at each vertex is parallel to the opposite side

12. The sides BC, CA, AB of a triangle touch an ellipse at P, Q, R, prove that $BP \cdot CQ \cdot AR = PC \cdot QA \cdot RB$

13. The tangents at the extremities of a chord PQ of an ellipse meet at T, if the eccentric angles of P, Q differ by a constant, find the locus of T

14 Prove that the area of the minimum triangle which can be described about an ellipse, semi axes a, b, is $3\sqrt{3}ab$

15. An ellipse cuts the sides BC, CA, AB of a triangle at P_1, P_2, Q_1, Q_2, R_1, R_2, prove that

$$BP_1 \cdot BP_2 \cdot CQ_1 \cdot CQ_2 \cdot AR_1 \cdot AR_2 = CP_1 \cdot CP_2 \cdot AQ_1 \cdot AQ_2 \cdot BR_1 \cdot BR_2$$

16. A line cuts two concentric homothetic ellipses at P, Q, H, K, prove that $PH = QK$

17 H, K are two fixed points on an ellipse, HP, KQ are two variable parallel chords, find the envelope of PQ

18 From a point on an ellipse, tangents are drawn to a concentric homothetic ellipse, touching it at P, Q and meeting the first ellipse at R, S, prove that $PQ = \frac{1}{2}RS$

19. Through a variable point O two lines of fixed direction are drawn cutting a given ellipse at P, Q and P′, Q′, prove that $\frac{OP \cdot OQ}{OP' \cdot OQ'}$ is constant What special value is obtained by taking O at the centre of the ellipse?

20. A triangle PQR is inscribed in an ellipse and the tangents at the vertices of the triangle meet the opposite sides at X, Y, Z; prove that X, Y, Z are collinear.

Diameters of an Ellipse. If the projection of the ellipse is the circle centre c and if pq is a variable diameter of the circle, then $pc = cq$. Consequently for the ellipse PC = CQ so that every chord through C is bisected at C; C is therefore the centre of the ellipse, and any line through the centre is called a *diameter* of the ellipse. Consequently the projection of a diameter of an ellipse is a diameter of the circle. Further, the mid-points of a system of parallel chords of an ellipse lie on a straight line, a diameter of the ellipse, because their projections are the mid points of a system of parallel chords of a circle. And if PCP′, QCQ′ are two diameters of an ellipse such that the mid-points of chords parallel to PCP′ lie on QCQ′, then the mid-points of chords parallel to QCQ′ lie on PCP′, for their projections are perpendicular diameters of the circle We then call PCP′, QCQ′ *conjugate diameters* of the ellipse, and we see that if the ellipse is projected into a circle, any pair of conjugate diameters project into diameters at right angles.

Poles and Polars. If a variable line is drawn through a fixed point P to cut an ellipse at H, K, and if Q is the harmonic conjugate of P w r t H, K, then the locus L of Q is called the *polar* of P w.r.t. the ellipse, and Q is called the pole of L

***Theorem 15.** (1) A point and its polar w.r t an ellipse project into a point and its polar w.r.t. the circle into which the ellipse is projected.

(2) The polar of a point w r t. an ellipse is a straight line.

The projection of a harmonic range is another harmonic range. This theorem therefore follows at once from the polar properties of a circle.

Theorem 16. (1) If the polar of P passes through Q, then the polar of Q passes through P.

[P, Q are called **conjugate points** w.r.t. the ellipse.]

(2) If the pole of the line HK lies on the line MN, then the pole of MN lies on HK.

[HK, MN are called **conjugate lines** w.r.t. the ellipse.]

(3) Conjugate points and conjugate lines w.r.t. an ellipse project into conjugate points and conjugate lines w.r.t. the circle into which the ellipse is projected.

(4) The centre of the ellipse is the pole of the line at infinity.

(5) Conjugate diameters of an ellipse are conjugate lines w.r.t. the ellipse.

These results are evident, by projection.

In order to illustrate the application of orthogonal projection, the following examples are added. The reader should note that lengths of lines are altered by projection; consequently metrical properties must be proved either by casting them into the form of *ratios* of segments of the same or parallel lines or by means of cross-ratios.

Example I. CP, CD are two conjugate semi-diameters of an ellipse; two other conjugate semi-diameters meet the tangent at P in H, K; prove that $HP \cdot PK = CD^2$.

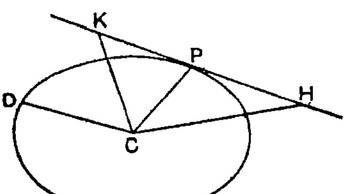

FIG. 7.

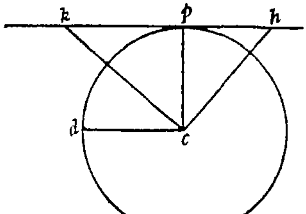

FIG. 8.

Project the ellipse into a circle; since *cp*, *cd* are conjugate diameters of the circle, $\angle pcd = 90°$; therefore *cd* is parallel to *hk*, and so CD is parallel to HK.

$\therefore$ each of the ratios $\frac{HP}{CD}, \frac{PK}{CD}$ is unaltered by projection.

Since ch, ck are conjugate diameters of the circle, $\angle hck = 90°$;

$$\therefore\ hp \,.\, pk = cp^2 = cd^2,$$

$$\therefore\ \frac{hp}{cd} \cdot \frac{pk}{cd} = 1;$$

$$\therefore\ \frac{\mathrm{HP}}{\mathrm{CD}} \cdot \frac{\mathrm{PK}}{\mathrm{CD}} = 1 \quad \text{or} \quad \mathrm{HP} \,.\, \mathrm{PK} = \mathrm{CD}^2. \qquad \text{Q.E.D.}$$

Example II. If a parallelogram circumscribes an ellipse, and if its sides are parallel to conjugate diameters, then its area is constant.

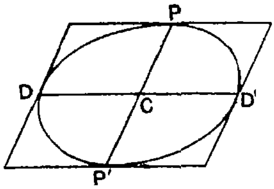

FIG. 9.

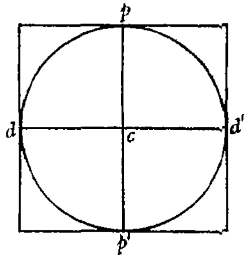

FIG. 10.

Let the semi-axes of the ellipse be a, b, project the ellipse into a circle, radius b, by projecting on to a plane making an angle θ with the plane of the ellipse, where $\cos\theta = \frac{b}{a}$. [Th 14.]

Conjugate diameters of the ellipse project into perpendicular diameters of the circle, and therefore the parallelogram becomes a square of side $2b$ and area $4b^2$.

$$\text{the area of the parallelogram} = \frac{4b^2}{\cos\theta} = 4b^2 \times \frac{a}{b} = 4ab.$$

Q.E.D.

EXERCISE II. b.

1 The centroid of the triangle PQR, inscribed in an ellipse, is at the centre of the ellipse If the ellipse is projected into a circle, prove that the projection of PQR is an equilateral triangle

2. If CP, CD are conjugate semi diameters of an ellipse, prove that the tangent at P is parallel to CD

3. If PQ is a chord of an ellipse and if the tangents at P, Q meet at T, prove that T is the pole of PQ

4. T is the pole of a chord PQ of an ellipse, centre C; prove that CT bisects PQ.

5. With the data of No. 4, if CT cuts PQ at V and the ellipse at H, prove that $CV \cdot CT = CH^2$.

6. Prove that the area of the maximum triangle that can be inscribed in an ellipse, semi-axes a, b, is $\frac{3\sqrt{3}}{4}ab$ and that the centroid of the triangle is at the centre of the ellipse.

7. P is any point on an ellipse, centre C, major axis ACA′; a line AQ parallel to CP cuts the ellipse at Q and the minor axis at R; prove that $AQ \cdot AR = 2CP^2$.

8. The tangents TP, TQ at the points P, Q on an ellipse are at right angles; PH, QK are the normal chords at P, Q; prove that

$$TP \cdot PH = TQ \cdot QK.$$

9. T is a variable point on a tangent to a given ellipse at a fixed point P; from the mid-point M of TP the other tangent MQ is drawn to touch the ellipse at Q; prove that TQ passes through a fixed point.

10. PP′ is a diameter of an ellipse; any chord P′D cuts the tangent at P in Q; prove that the tangent at D bisects PQ.

11. A tangent to an ellipse meets two conjugate diameters at T, T′, prove that the other tangents from T, T′ to the ellipse are parallel.

12. N is the foot of the perpendicular from a point P on an ellipse, centre C, to its major axis AA′; NQ is drawn parallel to AP to meet CP at Q; prove that AQ is parallel to the tangent at P.

13. Lines are drawn through the vertices of a triangle inscribed in an ellipse, parallel to the diameters bisecting the opposite sides; prove that these lines are concurrent.

14. Generalise the property known as Simson's Line.

15. CP, CD are conjugate semi diameters of an ellipse; PN, DR are the perpendiculars to the major axis; prove that (i) $PN^2 + DR^2 = CB^2$, (ii) $CN^2 + CR^2 = CA^2$, (iii) $CP^2 + CD^2 = CA^2 + CB^2$, (iv) $PN \cdot NC = DR \cdot RC$.

16. CP, CD are conjugate semi-diameters of an ellipse; PD cuts the major and minor axes at M, N; prove that $\frac{CA^2}{CM^2} + \frac{CB^2}{CN^2} = 2$.

17. If P, D are two points on an ellipse whose eccentric angles differ by a right angle, prove that CP, CD are conjugate semi diameters.

18. PQ is a diameter of the ellipse, R is any point on the curve; prove that the diameters parallel to PR, RQ are conjugate.

19. PCP′, DCD′ are the equal conjugate diameters of an ellipse, CP meets the tangent at the vertex A in H, prove that either PD^2 or PD'^2 equals $2AH^2$.

20. The tangents at the extremities P, P′ of a diameter of an ellipse meet any other tangent at H, K and any two conjugate diameters at L, M; prove PL . P′M = PH . P′K.

21. Generalise the property: if PQ is a diameter of a circle and if R is a variable point on the circumference, $PR^2 + RQ^2$ is constant.

22. T is the pole of a chord PQ of an ellipse, centre C; TP, TQ meet CQ, CP at Q′, P′; prove $\triangle$ TPP′ = $\triangle$ TQQ′.

23. T is the pole of a chord PQ of an ellipse; a chord HK parallel to TP meets PQ at V, TQ at R; prove $RV^2 = RK . RH$.

24. CP, CD are conjugate semi-diameters of an ellipse; G is the centroid of the sector PCD; GN is drawn parallel to CD to meet CP at N; prove $\dfrac{GN}{CD} = \dfrac{CN}{CP} = \dfrac{4}{3\pi}$. [If O is the centre and AC a chord of a circle and if G is the centroid of sector AOC, then $\dfrac{OG}{OA} = \dfrac{2}{3} \cdot \dfrac{\text{chord AC}}{\text{arc AC}}$.]

25. Find the position of the centroid of the area bounded by a diameter of an ellipse and the portion of the curve on one side of it.

26. Of all triangles that can be inserted in an ellipse, the triangle whose vertices have as eccentric angles ϕ, $\phi + \dfrac{2\pi}{3}$, $\phi + \dfrac{4\pi}{3}$ is of maximum area.

27. Generalise: the tangents from any point to a circle are equal.

28. Generalise: if a triangle PQR inscribed in a circle is such that the tangent at each vertex is parallel to the opposite side, the triangle is equilateral; and if T is any point on the circumference

$$TP \pm TQ \pm TR = 0.$$

29. A triangle, area Δ, sides a, b, c, is the orthogonal projection of an equilateral triangle; prove that the angle between the two planes depends only on the ratio $\dfrac{\Delta}{a^2 + b^2 + c^2}$.

30. ABCD is a rhombus of side 2 inches; $\angle$BAC = 60°; the rhombus is the orthogonal projection of a square in another plane. Prove that the side of the square is $\sqrt{6}$ inches, and find the angle between the planes.

31. Two adjacent sides of a parallelogram are of lengths a, b and are inclined at an angle ϕ; the parallelogram is projected into a square, side x; prove that $2x^2 = a^2 + b^2 - \sqrt{(a^4 + 2a^2b^2 \cos 2\phi + b^4)}$.

[Draw an ellipse to touch the sides of the parallelogram at their mid-points.]

32. The normals at the points P, Q of an ellipse, centre C, are perpendicular and meet the ellipse again at P′, Q′; prove that the sectorial areas CPP′, CQQ′ are equal.

CHAPTER III

CONICAL PROJECTION

THE idea of conical projection originated with Serenus (450 A.D.); but no real use was made of it before the time of Desargues (1593-1662), a French architect and engineer, who served under Richelieu at the siege of Rochelle. The modern theory of projective geometry is only a development of Desargues' work, and it is worth noticing that Von Staudt's non-metrical *Geometry of Position* takes as its starting-point Desargues' property of perspective triangles. Desargues' ideas were not, however, appreciated at their true value by his contemporaries, excepting Pascal, chiefly because the new analytical field of discovery, opened by Descartes, appeared more attractive and fruitful. It remained for Carnót (1753-1823), Poncelet (1788-1867) and Chasles (1793-1880) to perceive their merits and develop them. Poncelet by his conception of "the circular points at infinity" and his discovery of their connection with the foci of a conic and by his enunciation of the "Principle of Duality" was enabled to coordinate and generalise the theory of conics, elevating it from a collection of isolated theorems to a connected and logical unity. The analytical basis for Poncelet's method of reciprocal polars was supplied by Plucker by his invention (1829) of tangential coordinates.

Definitions.

(1) P_1, P_2, ... are a system of points in a plane Σ. O is any fixed point outside Σ; the lines OP_1, OP_2, . . meet a second given plane σ at the points $p_1, p_2, \ldots$. Then the system of points $p_1, p_2, \ldots$ is said to be the **conical projection** of the given system in Σ on σ w.r.t. the point O, which is called the **vertex of projection.**

(2) The line of intersection of Σ and σ is called the **axis of projection.**

(3) If a plane through O parallel to σ meets Σ in the line L, then L is called the **vanishing line** of Σ, similarly, if a plane through O parallel to Σ meets σ in the line m, then m is called the **vanishing line** of σ.

(4) The meet of any line in a plane with the vanishing line of that plane is called the **vanishing point** of the line

Note Unless otherwise stated, capital letters refer to elements of the given figure and small letters to the corresponding elements of the projected figure.

Theorem 17. (1) A straight line projects into a straight line.

(2) The meet of two straight lines projects into the meet of their projections.

(3) The join of two points projects into the join of their projections.

(4) Any point on the axis of projection projects into itself.

(5) Any line and its projection meet on the axis of projection.

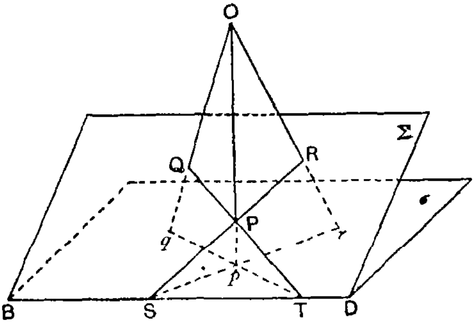

FIG 11.

The proof of these statements is evident from the figure.

Theorem 18. (1) The cross ratio of four collinear points is equal to the cross ratio of their projections: and in particular the projection of a harmonic range is a harmonic range.

(2) The cross ratio of four concurrent lines is equal to the cross-ratio of their projections and in particular the projection of a harmonic pencil is a harmonic pencil

The proof of these statements follows at once from the fundamental cross ratio property of a pencil.

Theorem 19. (1) If a system of concurrent lines in Σ meet at a point on the vanishing line of Σ, then their projections form a system of parallel lines in σ.

(2) Each point on the vanishing line projects into an ideal point or point at infinity in the projected system

(3) The vanishing line projects into an ideal line, the line at infinity, in the projected system.

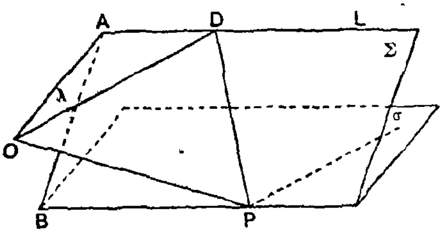

FIG 12

(1) Let Σ, σ be the two planes and O the vertex of projection. The vanishing line L of Σ is the intersection of Σ with a plane λ through O, parallel to σ. Let D be the point on L at which the given system of lines concur. Let any line DP of the system meet the axis of projection at P. Then, by definition, the projection of DP is the line of intersection of the planes ODP, σ

Now λ and σ are parallel planes ; $\therefore$ the plane ODP cuts λ, σ in parallel lines.

$\therefore$ the projection of DP is parallel to OD

$\therefore$ the given system of lines through D project into a system of lines parallel to OD. Q.E.D.

(2) Now a system of parallel lines determine an ideal point or point at infinity, common to each member of the system. Therefore the projection of the point D determined by the system of concurrent lines is the ideal point determined by the system of parallel lines into which they project. Q.E D.

(3) Further, the line at infinity is the aggregate of all ideal points in the plane, determined by all possible systems of parallel lines

But every point on the vanishing line projects into an ideal point. Therefore the vanishing line projects into the line at infinity.

Q. E. D.

Corollary. If D is the vanishing point of the line PD, and if O is the vertex of projection, the projection of PD is parallel to OD.

Theorem 20. If H, K are the vanishing points of two lines QH, QK, and if O is the vertex of projection, the angle between the projections of QH, QK is equal to ∠HOK.

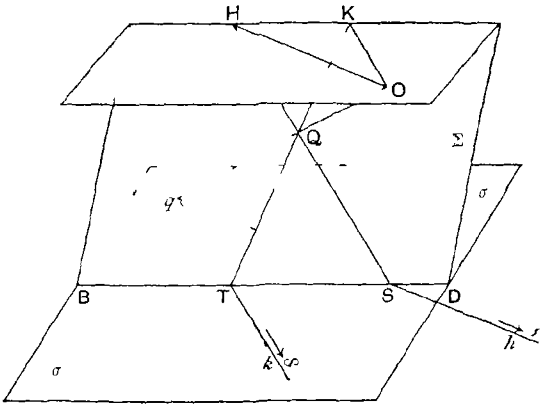

FIG. 13

Denoting elements of the projected figure by small letters we see that qh, qk are respectively parallel to OH, OK.

∴ $\angle hqk = \angle$HOK. Q. E. D.

Theorem 21. (1) Given any geometrical system in a plane Σ and a vertex of projection O, it is possible to find a plane σ such that any given line PQ in Σ projects into the line at infinity in σ.

(2) Given two angles HAK, PBQ, and a line CD in a plane Σ, it is possible to find a vertex of projection O and a plane of projection

σ such that the angles project into angles of given sizes, α, β respectively, and the line CD projects into the line at infinity in σ.

(1) Take for σ any plane parallel to the plane OPQ; then PQ is the vanishing line of Σ, and therefore projects into the line at infinity in σ.

(2) Let AH′, AK′, BP′, BQ′ meet CD at H, K, P, Q. Take any other plane through CD and describe in it on HK, PQ segments of

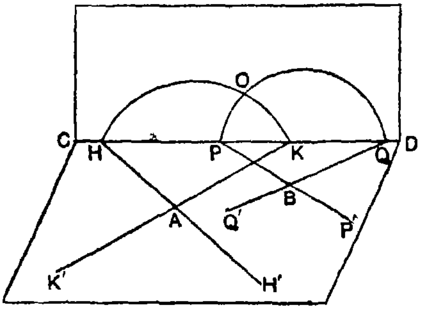

FIG. 14.

circles containing angles equal to α, β respectively, and let O be one of the points of intersection of these circles. With O as vertex, project the system on to any plane parallel to the plane OCD.

Then, by Theorem 20, since H, K, P, Q are vanishing points, it follows that the projected angles *hak*, *pbq* are equal to α, β respectively and the line CD is projected to infinity. Q.E.D.

Note. (i) The two circles may not cut at real points; in such a case the vertex of projection is imaginary. Since, however, the process of projection corresponds to a definite analytical transformation (see p. 40) it may still be regarded as valid. Any results obtained from such a process are valid even if the process has no graphical significance; the employment of geometrical language is merely a convenient means of describing a particular analytical operation.

(ii) We cannot distinguish between the magnitudes of angles *hak*, *hak′*; either of these may prove to be α, and the other will be its supplement.

(iii) Theorem 21 (2) may also be stated in the following form :

Given a triangle ABC and a line PQ in its plane, it is possible to project ABC into a triangle similar to a given triangle and at the same time the line PQ to infinity.

Theorem 22. [Desargues' Theorem.]

(1) ABC, A′B′C′ are two triangles in the same or different planes. If AA′, BB′, CC′ are concurrent, then the meets L, M, N of BC, B′C′; CA, C′A′ , AB, A′B′ are collinear

(2) *Conversely*, if L, M, N are collinear, then AA′, BB′, CC′ are concurrent.

(1) Suppose first that ABC, A′B′C′ lie in different planes Σ, Σ', and let AA′, BB′, CC′ concur at O

Then A′B′C′ is the projection of ABC w r t. O on Σ'

BC meets B′C′ on the axis of projection.

L lies on the line of intersection of Σ, Σ'.

Similarly M, N lie on this line , ∴ L, M, N are collinear

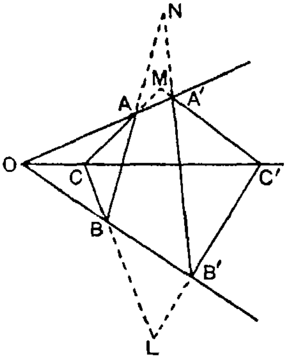

FIG 15

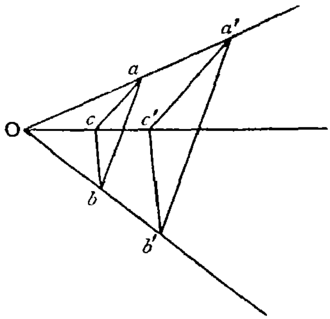

FIG 16

Suppose next that ABC, A′B′C′ lie in the same plane Project LM to infinity Since BC, B′C′ meet at L, *bc*, *b′c′* are parallel Since CA, C′A′ meet at M, *ca*, *c′a′* are parallel

$$\frac{Ob}{Ob'} = \frac{Oc}{Oc'} = \frac{Oa}{Oa'};$$

∴ *ab*, *a′b′* are parallel, and so AB, A′B′ meet on LM ,

∴ L, M, N are collinear. Q.E.D.

(2) For the converse, suppose first that ABC, A′B′C′ lie in different planes.

Since BC, B′C′ intersect, they lie in a plane, α say.

Similarly, let β, γ be the planes CC′AA′, AA′BB′.

Then AA′, BB′, CC′ are the lines of intersection of the pairs of planes β, γ ; γ, α ; α, β. But any three planes have one common point, so that their lines of intersection are concurrent ;

$\therefore$ AA′, BB′, CC′ are concurrent.

Suppose next that ABC, A′B′C′ lie in the same plane.

Project the line LMN to infinity ; then ab, bc, ca are parallel to $a'b'$, $b'c'$, $c'a'$, and the triangles abc, $a'b'c'$ are similar. Suppose $a'a$, $c'c$ meet at O_1 and $b'b$, $c'c$ meet at O_2

Then $$\frac{O_1c}{O_1c'} = \frac{ca}{c'a'} = \frac{cb}{c'b'} = \frac{O_2c}{O_2c'} ;$$

$\therefore$ O_1 coincides with O_2, and so aa', bb', cc' are concurrent.

Q.E.D.

Definition. If the joins of corresponding vertices of two triangles are concurrent, the triangles are said to be **in perspective** ; the point of concurrency is called the **centre of perspective**, and the line on which corresponding sides intersect is called the **axis of perspective.**

Theorem 23. If a line APB touches a curve S at P, then the projection of APB touches the projection of S at the projection of P.

The tangent at P is the limiting position of the chord PQ when Q tends towards P along the curve ; and the tangent at p is the limiting position of pq when q tends to p.

$\therefore$ the tangent at p is the projection of the tangent at P.

Q.E.D.

Example. To establish the harmonic property of the quadrilateral.

With the notation of Fig. 17, it is required to prove that {AECP} is harmonic.

Project FG to infinity ; Fig. 18 represents the projection.

AD, CB meet at F ; $\therefore$ ad, cb are parallel.

DC, AB meet at G ; $\therefore$ dc, ab are parallel.

abcd is a parallelogram, $ae = ec$

Since P lies on FG, p is a point at infinity.

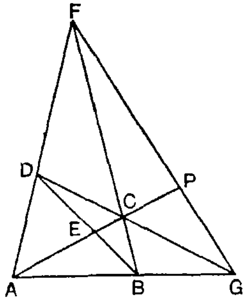

FIG. 17.

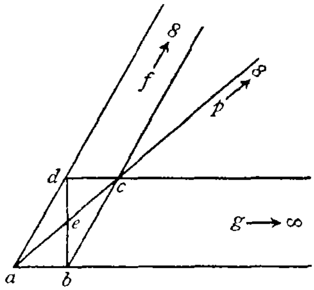

FIG. 18.

Since $ae = ec$, $\{aecp\}$ is harmonic.

$\{AECP\}$ is harmonic. Q E D.

EXERCISE III. a.

1. A, B, C are three collinear points in order; show how to project so that c is the mid point of ab.

2. A, B, C are three collinear points; show how to project so that the ratio $\frac{ab}{bc}$ equals any given value.

3. Given a triangle ABC and any point P in its plane, show how to project so that p is the centroid of triangle abc.

4. Prove that the vanishing lines in Σ and σ are parallel.

5. A straight line meets the sides BC, CA, AB of a triangle ABC at P, Q, R; P′, Q′, R′ are the harmonic conjugates of P, Q, R w.r.t. BC, CA, AB respectively; prove that AP′, BQ′, CR′ are concurrent.

6. Generalise: the mid points of the three diagonals of any quadrilateral are collinear.

7. P, Q, R are points on the sides BC, CA, AB of a triangle; prove that $\frac{BP}{PC} \cdot \frac{CQ}{QA} \cdot \frac{AR}{RB}$ is unaltered by projection.

Deduce Menelaus' Theorem from this result.

8. A, B, C, D, E, F are six collinear points; l, m, n are any constants such that $l \cdot AB \cdot CD \cdot EF + m \cdot AC \cdot BE \cdot DF + n \cdot AD \cdot BF \cdot CE = 0$; prove

that this relation is unaltered by projection [This is a generalisation of the fundamental cross-ratio property]

9. AB is the axis of projection, CD is the vanishing line of Σ, $A'B'$ is the reflection of AB in CD ; prove that the length of any segment of $A'B'$ is unaltered by projection

10. A, B, C , A', B', C are two sets of three collinear points , prove that the meets of AB', $A'B$, BC', $B'C$; CA', $C'A$ are collinear.

11. A, B, C are three fixed collinear points , PQR is a variable triangle such that P, Q lie on fixed lines and QR, RP, PQ pass through A, B, C respectively ; prove that the locus of R is a straight line

12. Any line meets the sides AD, DC, CB, BA of a quadrilateral at P, Q, R, S ; P', Q', R', S are the harmonic conjugates of P, Q, R, S w.r.t AD, DC, CB, BA , prove that PQ, $P'Q'$, R S' are concurrent

13 Three triangles are such that their vertices lie on the same three concurrent lines , prove that the axes of perspective of the triangles taken in pairs are concurrent

14. P is a variable point in Σ ; H, K are two fixed points on the vanishing line of Σ , prove that $\angle hpk$ is constant

15. Show how any quadrilateral can be projected into a square of given size.

16 Show how two quadrilaterals which have a common third diagonal can be projected into rhombuses

17 (i) ABCD are four collinear points , P, Q are a pair of points harmonically conjugate to B, C and to A, D , prove that the locus of points at which AB, CD subtend equal angles is the circle on PQ as diameter

(ii) Show how to project any three angles into three equal angles and at the same time any line to infinity

18. Given the planes Σ, σ and the vertex of projection O, show that there are two points M, N in Σ such that any angle in Σ whose apex is at M or N is unaltered in size by projection

[Let XY be the axis of projection ; let α, β be the planes through X, Y which bisect the angles between Σ, σ , from O draw perpendiculars to α, β meeting Σ in M, N and σ in m, n , take any two points A, B on XY, and show that $\angle AMB = \angle AmB$]

Analytical Treatment of Conical Projection.

V is the vertex and BD the axis of projection. The plane through V parallel to σ meets Σ in the vanishing line AC. The plane through V perpendicular to BD meets BD, AC at B, A P is any point in Σ, VP meets σ at p.

Take as origin any point O on BD · for axes, in Σ take Ox, Oy along and perpendicular to OD, and in σ take $O\xi$, $O\eta$ along and

perpendicular to OD. Let (x, y) and (ξ, η) be the coordinates of P, p referred to these axes.

Let $VA = a$, $AB = b$, $BO = c$.

Let AP meet BD at m; since the plane VPA cuts the parallel planes VAC, σ in parallel lines, VA is parallel to pm; but VA is

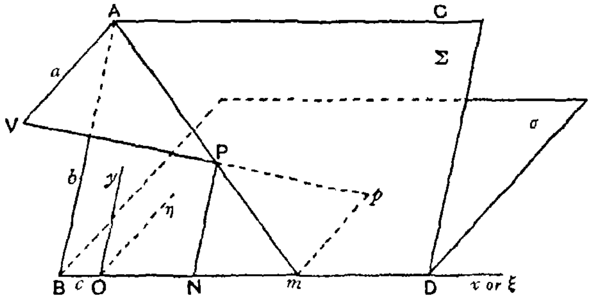

FIG 19.

perpendicular to BD since the plane VAB is perpendicular to BD; therefore pm is perpendicular to BD, $pm = \eta$, $Om = \xi$.

Draw PN perpendicular to BD, so that $PN = y$, $ON = x$.

By parallels,
$$\frac{pm}{VA} = \frac{Pm}{PA} = \frac{Nm}{NB};$$

$$\therefore \frac{\eta}{a} = \frac{\xi - x}{x + c} \quad \text{or} \quad \eta x + \eta c = a\xi - ax;$$

$$\therefore x(\eta + a) = a\xi - c\eta \quad \text{or} \quad x = \frac{a\xi - c\eta}{\eta + a}.$$

Similarly,
$$\frac{PN}{AB} = \frac{mP}{mA} = \frac{mp}{mp + VA};$$

$$\therefore \frac{y}{b} = \frac{\eta}{\eta + a} \quad \text{or} \quad y = \frac{b\eta}{\eta + a}.$$

Therefore, by a proper choice of axes, any figure and its conical projection are related by the equations

$$x = \frac{a\xi - c\eta}{\eta + a}; \quad y = \frac{b\eta}{\eta + a}.$$

If the origin is taken at B, $c=0$, and we obtain the simpler form,

$$x=\frac{a\xi}{\eta+a}; \quad y=\frac{b\eta}{\eta+a}.$$

These equations represent an analytical transformation which corresponds to the geometrical operation of conical projection, and therefore justify the process under all circumstances. The constants a, b, c may be regarded as defining the position of the vertex of projection: if a, b, c are all real, the vertex of projection is real; but if any of these constants are imaginary, V is an imaginary point, but this does not invalidate the process, viewed as an analytical operation. By an extension of the meaning of the terms employed, it is possible to indicate in geometrical form an analytical process which may have no graphical analogue, and in this way a laborious piece of analysis may be avoided. Any two conics can, for example, be projected into two circles by a *real* projection if they do not cut at real points; consequently any descriptive property connecting the two circles can be transformed into a property of the two conics by purely geometrical methods: but this transference could be effected with equal validity, although with less ease, by analysis If, however, the two conics have four real points of intersection, it is impossible to obtain a *real* projection which will change both of them into circles, but a transformation with imaginary coefficients can be found which will change both into circles, and so the analysis makes no distinction between the two cases, as the same analytical method applies whether the coefficients are real or imaginary, and therefore the transmitted property must still hold good. The analysis is indeed an essential part of the proof; but since it leads inevitably to a result that can be predicted, it may be omitted; the nature of the analysis is indicated by the geometrical description of the method by which the projection is to be effected.

Theorem 24. (1) Any conic can be projected into a circle

(2) Any curve formed by the projection of a circle is a conic.

(3) One and only one conic can be drawn through five given points, no four of which are collinear.

(1) This has already been proved in Theorem 3, p. 11, because

the homographic transformation there chosen is identical with the transformation established for conical projection on p 42

(2) If we apply the transformation $x=\dfrac{a\xi}{\eta+a}$, $y=\dfrac{b\eta}{\eta+a}$ to the general equation of a circle $x^2+y^2+2gx+2fy+c=0$, we obtain

$$a^2\xi^2+b^2\eta^2+2ga\xi(\eta+a)+2fb\eta(\eta+a)+c(\eta+a)^2=0,$$

which is of the second degree, and is therefore a conic

(3) The general equation of the second degree contains five **independent** constants, which enter linearly, they can therefore be chosen so as to make the curve pass through five given points, and this, in only one way, if no four of the points are collinear.

Q E D

Corollary. (1) Any curve formed by the projection of a conic is a conic

(2) The degree of any curve is unaltered by projection

Although in actual applications it is unnecessary to supply the analysis which is the actual justification of the process yet the reader may obtain a clearer idea of what is happening if some examples of the analytical working are given We shall use the notation of Fig 19

Example 1. Lines which intersect on the vanishing line project into parallel lines

The vanishing line AC is $y=b$, any system of lines intersecting on $y=b$ may be written $px+qy+\lambda(y-b)=0$ where λ varies

Putting $x=\dfrac{a\xi}{\eta+a}$, $y=\dfrac{b\eta}{\eta+a}$, we obtain

$$pa\xi+qb\eta-\lambda ab=0,$$

which represents a system of parallel lines, when λ varies

Example 2. A system of concentric circles project into a system of conics having double contact with each other, and the line at infinity projects into the chord of contact

Any circle of the system can be represented by

$$(x-f)^2+(y-g)^2-r^2=0,$$

where f, g are constant and r varies

Therefore its projection is

$$[a\xi - f(\eta + a)]^2 + [b\eta - g(\eta + a)]^2 - r^2(\eta + a)^2 = 0.$$

This represents a system of conics touching each other at their meets with $\eta + a = 0$, which is the projection of the line at infinity.

EXERCISE III. b.

1. Show that a system of conics having double contact with each other at two fixed points E, F can be projected into a system of concentric circles, and that in this case EF is projected into the line at infinity

[Take EF as $y - b = 0$, write the conics in the form

$$px^2 + 2qxy + ry^2 - \lambda(y - b)^2 = 0,$$

and use $x = \dfrac{a\xi - c\eta}{\eta + a}$, $y = \dfrac{b\eta}{\eta + a}$]

2. With the data of No 1, show that the projection is real if and only if $pr - q^2$ is positive, and that this requires that E and F should be imaginary

3. Show that a system of conics through four fixed points can be projected into a system of coaxal circles

[Take the conics in the form $px^2 + 2qxy + ry^2 + \lambda(y - b)(lx + my + 1) = 0$, where λ varies]

4. If a conic touches the vanishing line, prove that its projection is a parabola

5. Prove that a system of conics having the same focus and directrix can be projected into a system of concentric circles

6. If the conical projection is chosen so that $\xi = \dfrac{bx}{y - b}$, $\eta = \dfrac{ify}{y - b}$, show that any conic passing through (f, b) and $(-f, b)$ in the x, y plane projects into a circle.

7. Find the relations for a conical projection such that both

$$2x^2 + 3y^2 = 20 \quad \text{and} \quad x^2 = 2y$$

project into circles [Use No 6]

8. Repeat No 7 for the conics $x^2 - 2y + 1 = 0$, $2x^2 - y^2 = 2y - 1$, and show that the circles obtained are concentric. [Use No 6]

9. (i) Ox, Oy, Oz are three mutually perpendicular axes of reference any plane cuts them in A, B, C, and P is a point in this plane whose areal coordinates referred to $\triangle$ABC are ξ, η, ζ · if the coordinates of P referred to Ox, Oy, Oz are x, y, z, and if OA $= a$, OB $= b$, OC $= c$, prove that $x = a\xi$, $y = b\eta$, $z = c\zeta$.

(2) If any other plane cuts Ox, Oy, Oz at A′, B′, C′, and if P′ is the projection of P w r t O on this plane, and if ξ, η, ζ' are the areal co ordinates of P w r t Δ A B C′, and if OA $= a'$, OB $= b$, OC $= c$, prove that

$$\frac{\xi}{\frac{a'}{a}\xi'} = \frac{\eta}{\frac{b'}{b}\eta} = \frac{\zeta}{\frac{c'}{c}\zeta'}.$$

10. (i) If with the notation of No 9 (1), P_1 is the projection of P w r t. O on the plane $z = 1$, and if $a - b = c = 1$, show that the coordinates of P_1 are $(X_1, Y_1, 1)$, where $\frac{X_1}{\xi} = \frac{Y_1}{\eta} - \frac{1}{\zeta}$

(ii) What is the connection between the curve $x^2 + y^2 = 1$ (Cartesians) and $\xi^2 + \eta^2 = \zeta^2$ (areals) ?

(iii) What is the connection between the curve $f(x, y, 1) = 0$ (Cartesians) and $f(\xi, \eta, \zeta) = 0$ (areals) ?

CHAPTER IV

THE CONIC

THE discovery of the conic is attributed to Menaechmus (350–330 B.C.), a disciple of Plato, and was employed by him to solve the famous Delian problem, the Duplication of the Cube; but his researches were very limited, being probably restricted to the barest elements of the parabola $y^2=ax$ and the rectangular hyperbola $xy=c^2$, with its asymptotes. The earliest writer known to have regarded the conic as a section of a cone was Aristaeus (320 B.C.), while the first systematic treatment was given by Euclid (323–284 B.C.) in a book now lost. This formed the basis of the famous Κωνικά of Apollonius (247–205 B.C.), which gained him among the ancients the title of the "Great Geometer." It contains a remarkably complete account of the non-focal properties of the conic, its conjugate diameters and asymptotes, and includes the harmonic property of the pole and polar for the case in which the pole lies outside the curve—a theorem which was completed only after the lapse of eighteen centuries by Desargues. The sense of Continuity which Kepler had introduced, illustrated, for example, by his view that the parabola has a centre which is a point at infinity, was developed by the genius of Desargues. By using the idea of the line at infinity and the cognate notion of parallelism, he showed that the asymptotes could be regarded as tangents at infinity, the centre as the pole of the line at infinity, and conjugate diameters as a special case of conjugate lines: to him is also due the harmonic theory of the quadrangle inscribed in a conic.

The principal value of the process of projection is the link it supplies between a circle and a conic, thus affording a rapid means of generalising a large group of properties of the circle by transmitting them to the conic. Much of the power of this method is due to the cross ratio theory of the conic which is identified with the name of the famous French geometer, Chasles (1793-1880). It is remarkable that the fundamental theorem (Theorem 30), which is a simple deduction from Pappus' theorem (p 99) and Apollonius' theorem on the cross ratio of four concurrent lines should have remained unnoticed for another two thousand years. The method adopted in the present chapter is due to Chasles.

We started by defining the conic as any curve represented by an equation of the second degree (p 5). We then proved (Theorem 24, p 42) that any conic can be regarded as the projection of a circle, and that the projection of any circle is a conic. We shall now examine different types of projection, treating the conic as the projection of a given circle.

There are three kinds of conics.

(1) The conic is called a *hyperbola* if the line at infinity cuts it at real distinct points.

For a real vertex of projection this case is obtained by choosing a line which cuts the generating circle at real distinct points as the vanishing line.

(2) The conic is called a *parabola* if the line at infinity touches the conic.

For a real vertex of projection this case is obtained by choosing a tangent to the generating circle as the vanishing line.

(3) The conic is called an *ellipse* if the line at infinity cuts the conic at imaginary distinct points.

For a real vertex of projection this case is obtained by choosing a line which cuts the generating circle at imaginary distinct points as vanishing line.

These definitions may be put more concretely as follows.

Imagine a cone whose base is a circle and vertex O, let a plane L cut the cone, then the curve in which the cone cuts L is a conic.

If the plane through O, parallel to L, meets the base-circle in imaginary points (Fig. 20), the conic is a closed curve, and is called an *ellipse*.

If the plane through O, parallel to L, touches the base-circle, the conic is an open curve with one branch, and is called a *parabola*.

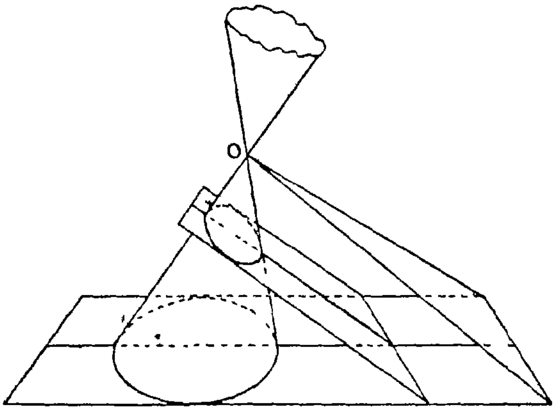

FIG 20

If the plane through O, parallel to L, meets the base circle at real points, the conic is an open curve with two branches (one on each half of the double-cone), and is called a *hyperbola*

For convenience of reference we shall repeat certain definitions which have already been given with special reference to the ellipse but hold generally.

Definition. If a variable line is drawn through a fixed point P meeting a fixed conic Σ at H, K, and if Q is the harmonic conjugate of P w.r.t. H, K, then the locus of Q is called the **polar** of P w r t Σ and P is called the **pole** of the locus of Q

Theorem 25. (1) If the conic Σ is the projection of the circle σ any pole and polar w.r't Σ is the projection of a pole and polar w.r.t. σ.

(2) The polar of a point P w r.t. a conic Σ is a straight line which

passes through the points of contact, real or imaginary, of the tangents from P to Σ.

(1) This is true because the projection of a harmonic range is a harmonic range.

(2) Since the polar of a point w.r.t. a circle is a straight line, it must also by projection be a straight line for a conic. Further, if a line through P is drawn to touch the conic at H, by definition the point on PH which belongs to the polar locus is H, if we regard the tangent as the limit of a secant. If P lies inside the conic H is imaginary, but the argument, expressed analytically, remains valid.

Q.E.D.

Theorem 26. (1) If the polar of a point P w.r.t. a conic passes through Q, then the polar of Q passes through P.

(2) If the pole of a line p lies on a line q, then the pole of q lies on p.

This follows at once by projection from a circle.

Definitions.

(1) Two points such that the polar of either w.r.t. a conic passes through the other are called **conjugate points** w.r.t. the conic.

(2) Two lines such that the pole of either w.r.t. a conic lies on the other are called **conjugate lines** w.r.t. the conic. If two chords of a conic are conjugate lines, they are called **conjugate chords** of the conic.

(3) If A, B, C are the poles of the sides of the triangle PQR w.r.t. a conic, the triangles ABC, PQR are called **conjugate triangles** w.r.t. the conic.

(4) If the vertices of a triangle are the poles of the opposite sides w.r.t. a conic, the triangle is called a **self-conjugate triangle** w.r.t. the conic.

Theorem 27. Two conjugate lines intersecting at a point T are harmonically conjugate to the tangents from T to the conic.

Conversely, two lines which are harmonically conjugate to the tangents from a point T to a conic are conjugate lines w.r.t. the conic.

This follows at once by projection from a circle, or may be deduced from Theorem 26.

Theorem 28. The cross-ratio of the range formed by four collinear points is equal to the cross-ratio of the pencil formed by their polars w.r.t. any conic.

This follows at once by projection from a circle.

Theorem 29. T is the pole of a chord AB of a conic; the tangent at any other point C meets TA, TB, AB at H, K, D; then {HK; CD} is harmonic.

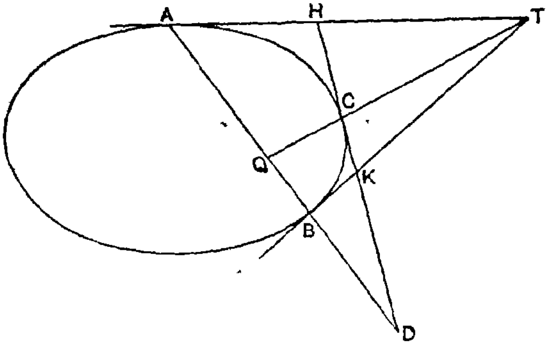

FIG 21

Let TC cut AB at Q.

The polar of T passes through D, therefore the polar of D passes through T; but the polar of D passes through C, and is therefore TC

∴ {AB; QD} is harmonic, and so T{AB; QD} is harmonic.

∴ {HK; CD} is harmonic Q.E.D.

EXERCISE IV. a.

1. P, Q are two conjugate points w r t a conic, R is the pole of PQ, prove that PQR is a self conjugate triangle
What is the dual theorem?

2. A variable chord PQ of a conic passes through a fixed point, prove that the tangents at P, Q meet on a fixed line

3. T is a variable point on a fixed line; TA, TB are the tangents to a given conic; prove that AB passes through a fixed point

4 D is a given point on the base BC of a fixed triangle ABC; prove that the polar of D w r t a variable conic touching AB, AC at B, C is a fixed line.

5 T is the pole of a chord PQ of a conic, if the portion of another tangent intercepted by TP, TQ is bisected at its point of contact, prove that it is parallel to PQ

6 QR is a fixed chord of a conic, P is a variable point on the conic, prove that the harmonic conjugate of the tangent at P w r t PQ, PR passes through a fixed point

7 The hypotenuse BC of a right angled triangle ABC cuts a conic at H, K, if B is the pole of AC, prove that AC bisects ∠HAK

8 PQR is a self conjugate triangle w r t a conic, with P inside the conic, prove that the chord through P parallel to QR is bisected at P

9 OA, OB are two chords of a conic equally inclined to the tangent O, prove that the pole of AB lies on the normal at O

10 T is the pole of a chord PQ of a conic, the bisector of ∠PTQ meets PQ at H, RS is any chord through H, prove that TH bisects the angle RTS

11 ABCD is a quadrangle AB, CD meet at E AC, BD meet at G, prove that a conic touching AD, BC at A, B and passing through G will touch EG.

12 Three conics are drawn through a common point D to touch AB, AC at B, C, BC, BA at C, A CA CB at A, B, prove that the tangents at D to the conics meet BC, CA, AB respectively in three collinear points

13 Two tangents to a conic at A, B meet at right angles at D, the tangent at any other point P meets AB, AD at Q, R, prove that DR bisects ∠PDQ

14 T is the pole of a chord PQ of a conic, the bisector of ∠PTQ meets PQ at H, prove that the pole of any other chord through H lies on the other bisector of ∠PTQ

15 B, C are conjugate points w r t a conic S, P is any point on S, BP, CP cut S again at Q, R, prove that QR passes through the pole of BC

16 AC, BD are chords of a conic such that the pole N of AC lies on BD AD meets the tangent at B in L, AB meets the tangent at D in M, prove that L M, N are collinear

17 The tangent at a point P on the circle whose diameter is the minor axis of an ellipse cuts the ellipse at Q, R, the tangents at Q, R meet in T, prove that PT is parallel to the major axis

18 A variable plane through a fixed line L cuts a fixed cone in the conic σ; prove that the locus of the pole of L w r t σ is a straight line.

Theorem 30. [Chasles' Theorem.] A, B, C, D are four fixed points on a conic; a, b, c, d are the tangents to the conic at these points. P is a variable point on the conic; t is a variable tangent to the conic. Then the cross-ratio of the pencil P{ABCD} is constant, and the cross-ratio of the range $t\{abcd\}$ is constant, and the values of these two cross-ratios are equal.

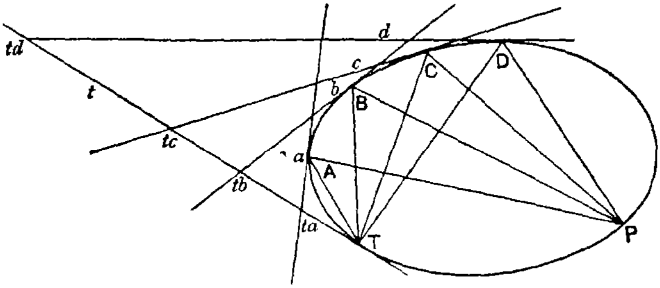

FIG. 22

Regard the conic as the projection of a circle and denote by dashes corresponding points in the plane of the circle: the latter is not shown in the figure.

Let T be the point of contact of t with the conic.

In the plane of the circle, the pencils P′{A′B′C′D′}, T′{A′B′C′D′} are equiangular, *i.e.* corresponding angles are either equal or supplementary; therefore the pencils are equicross.

$\therefore$ P{ABCD} = T{ABCD};

$\therefore$ P{ABCD} is of constant ratio. Q.E.D.

Again, the pole of TA is the meet of t, a, *i.e.* the point ta.

$\therefore$ by Theorem 28, the range $t\{abcd\}$ is equicross with the pencil

T{ABCD} = P{ABCD} = constant. Q.E.D.

Definitions.

(1) If A, B, C, D are four points on a conic such that P{ABCD} is harmonic, where P is any other point on the conic, then A, B, C, D are called **a harmonic system of points** on the conic.

(2) If a, b, c, d are four tangents to a conic such that $p\{abcd\}$ is a harmonic range, where p is any other tangent to the conic, then a, b, c, d are called a **harmonic system of tangents** to the conic.

Theorem 31. (1) If AC, BD are conjugate chords of a conic, then A, B, C, D form a harmonic system of points on the conic.

(2) Conversely, if A, B, C, D form a harmonic system of points on the conic, then AC, BD are conjugate chords of the conic.

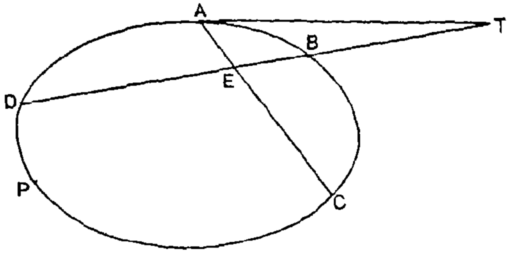

FIG 23

(1) Let the tangent at A meet BD at T and let P be any point on the conic. Then the pole of AC lies both on BD and AT, and is therefore T

$$\therefore \{TBED\} \text{ is harmonic.}$$

$$\therefore P\{ABCD\} = A\{ABCD\} = A\{TBED\},$$

$$\therefore P\{ABCD\} \text{ is harmonic.} \qquad \text{Q.E.D.}$$

(2) With the same construction, we have

$$P\{ABCD\} = A\{ABCD\} = A\{TBED\}.$$

But P{ABCD} is harmonic, $\therefore$ {TBED} is harmonic

$\therefore$ the polar of T passes through E, but it also passes through A, and is therefore EA.

$$\therefore \text{T is the pole of AC, but T lies on BD.}$$

$$\therefore \text{AC, BD are conjugate chords.} \qquad \text{Q.E.D.}$$

Notes on Method. The following considerations will often suggest a way in which to look for the solution of a given problem, but the application of some of the following remarks will not be understood until a further portion of the chapter has been read.

(i) If P, Q, R are collinear points, and if it is required to prove that $PQ = QR$, it may be simplest to show that $\{PQR\infty\}$ is harmonic, where ∞ denotes the point at infinity on PQ; and more generally $\frac{PQ}{QR}$ is constant if $\{PQR\infty\}$ is of constant cross-ratio; and

$$PQ \cdot PS = PR^2 \quad \text{or} \quad \frac{PQ}{PR} = \frac{PR}{PS} \quad \text{if} \quad \{QPR\infty\} = \{RPS\infty\}.$$

(ii) To prove two or more lines parallel, it may be useful to employ the idea that they concur at a point at infinity or that their poles w.r.t. some conic lie on a diameter.

(iii) To prove a range is harmonic, it is sometimes possible to connect it by a pencil with a harmonic system of points on a conic, which are formed by any two conjugate chords.

(iv) For properties of asymptotes of a hyperbola, draw the generalised figure in which the asymptotes appear as tangents with the line at infinity as the chord of contact.

Similarly the generalised figure of a parabola is a conic touching a straight line which corresponds to the line at infinity.

These generalised figures may be regarded as projections of the actual figure with the line at infinity projected into a finite line. If the property is of a projective nature, the new figure indicates a general theorem of which the given property is a particular case. But the general theorem may be easier to establish, because the irrelevant details of the figure have been removed.

EXERCISE IV. b.

1. P, Q are two conjugate points w.r.t. a conic; prove that the tangents from P, Q to the conic form a harmonic system of tangents to the conic.

2. ABC is a triangle inscribed in a conic; a chord PQ meets BC, CA, AB at L, M, N; the tangent at P meets BC at T; prove that

$$\{TLBC\} = \{PQNM\}.$$

3. PQ, PQ′ are two chords of a conic equally inclined to the normal chord PP′ at P; prove that P′Q, P′Q′ are harmonically conjugate w.r.t. P′P and the tangent at P′.

4. O is the pole of a chord BC of a conic; OQP is a line cutting the conic at P, Q; BA is a chord parallel to OP; prove that AC bisects PQ.

5. PP′, QQ′ are conjugate chords of a conic; a line through P parallel to the tangent at Q cuts QQ′, QP′ at H, K; prove that PH = HK.

6. ABCDE are five points on a given circle, prove that

$$A\{BCDE\} = \frac{BC \cdot DE}{BE \cdot DC}.$$

7. PP′, QQ′ are conjugate chords of a conic; a line through P′ parallel to QQ′ cuts the conic at R; PR cuts QQ′ at H, prove that QH = HQ′.

8. ABC is a triangle inscribed in a conic; T is the pole of AB; any line through T cuts BC, AC at M, N; prove that M, N are conjugate points w.r.t. the conic.

9. PQ is a chord of a conic; O is any point on PQ; M is any point on the polar MN of O, a parallel through O to MQ cuts MP, MN at B, A; prove that OB = BA.

10. *OP, OP′ are two chords of a conic equally inclined to the chord OB; OA is the chord perpendicular to OB, prove that PP′ passes through the pole of AB.*

11. T is the pole of a chord MN of a conic; a chord PQ of the conic parallel to TN meets TM, MN at H, R; prove that $HR^2 = HP \cdot HQ$.

12. PQ is a chord of a conic bisecting another chord AB at O; the tangents at P, Q meet AB at S, T; prove that AS = BT.

13. A conic touches the sides of a quadrilateral. Deduce from Chasles' Theorem a property by taking the variable tangent as coinciding successively with the sides of the quadrilateral

14. Two parallel tangents touch a conic at P, P′; two other tangents cut them at Q, Q′ and R, R′; prove that PQ . P′Q′ = PR . P′R′. [Use the idea of No. 13.]

15. T is the pole of a chord MN of a conic; any line cuts the conic at P_1, P_2 and TM, TN, MN at Q_1, Q_2, R; prove that

(i) $\{Q_1P_1P_2R\} = \{RP_1P_2Q_2\}$,

(ii) $Q_1P_1 \cdot P_2R \cdot RQ_2 = -Q_2P_2 \cdot P_1R \cdot RQ_1$.

16. A, B, C, D, P are five points on a conic; α, β, γ, δ are the lengths of the perpendiculars from P to AB, BC, CD, DA; prove that

$$P\{ABCD\} = \frac{\alpha \cdot \gamma}{\beta \cdot \delta} \cdot \frac{AB \cdot CD}{AD \cdot CB};$$

hence prove *Pappus'* theorem that if A, B, C, D are fixed and if P varies $\frac{\alpha \cdot \gamma}{\beta \cdot \delta}$ is constant.

17. [Desargues' Theorem.] ABCD is a quadrangle inscribed in a conic; a line cuts the conic at P, P′ and AD, BC, AC, BD at Q, Q′, R, R′; prove that $\{PQRP'\} = \{PR'Q'P'\} = \{P'Q'R'P\}$.

18. ABC, PQR are two triangles inscribed in a conic; PQ, PR cut BC at Q′, R′; AB, AC cut QR at B′, C′; prove that

$$(RB'C'Q) = (R'BCQ').$$

Central Properties. When a circle Σ is projected into a conic σ, the pole of the vanishing line w.r.t. Σ is projected into the pole of the line at infinity w.r.t. σ.

Theorem 32. If C is the pole of the line at infinity w.r.t. a conic, then every chord through C is bisected at C.

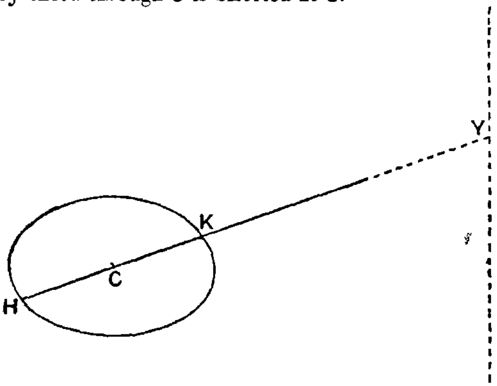

FIG. 24

Let HCK be any chord meeting the line at infinity at the ideal point Y. Then {HCKY} is harmonic

$\therefore$ HC = CK. Q.E.D.

Note. Ideal elements in a figure will always be represented by dots.

Definitions.

(1) The pole of the line at infinity w r.t a conic is called the **centre** of the conic

(2) Any chord through the centre of a conic is called a **diameter.**

(3) The tangents to a conic from its centre are called the **asymptotes** of the conic.

(4) Conjugate lines through the centre of a conic are called **conjugate diameters.**

(5) A chord of a conic which is conjugate to a diameter is called a **double ordinate** to that diameter

(6) A hyperbola, whose asymptotes are at right angles, is called a **rectangular hyperbola.**

Theorem 33. The mid-points of a system of parallel chords of a conic lie on a diameter.

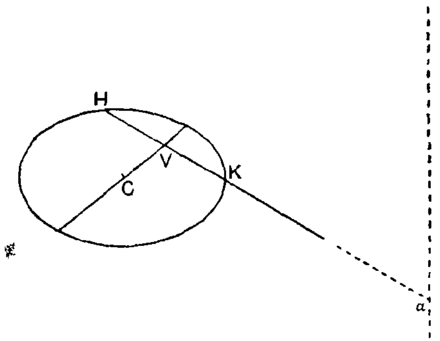

FIG 25

Let HK be any one of the system of parallel chords, which concur at the fixed ideal point α, let V be the mid point of HK

Since $\{HVK\alpha\}$ is harmonic, the polar of α passes through V

$\therefore$ V lies on a fixed line, viz the polar of α, which is a diameter since α is a point on the line at infinity. Q E.D.

Theorem 34. (1) If PCP′, DCD′ are two conjugate diameters of a conic, then PCP′ bisects all chords parallel to DCD′.

(2) If HK is any double ordinate to PCP′ meeting it at N, then HK is bisected at N and is parallel to the tangents at P, P′ to the conic and to DCD′.

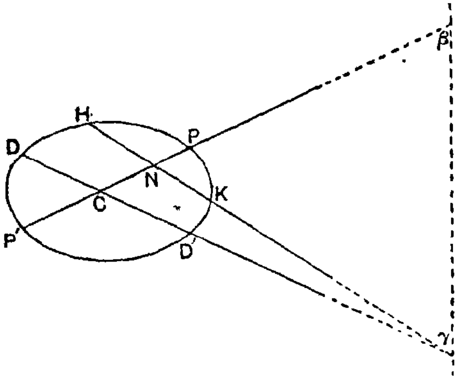

FIG. 26.

(1) Let PCP′, DCD′ meet the line at infinity at β, γ. Let HK be any chord parallel to DCD′; it therefore passes through γ

Now $C\beta$ contains the pole C of $\beta\gamma$; therefore $\beta\gamma$ contains the pole of $C\beta$; but by hypothesis $C\gamma$ contains the pole of $C\beta$.

$$\therefore\ \gamma \text{ is the pole of } C\beta;$$

$$\therefore\ \{HNK\gamma\} \text{ is harmonic, and so } HN = NK. \qquad \text{Q.E.D.}$$

(2) Let any line through γ, the pole of $C\beta$ or CP, cut the conic at H, K; then HK is a double ordinate to CP. But by (1), HK is bisected by CP.

Further, since γ is the pole of PP′, γP and γP′ are the tangents at P and P′. Therefore HK, CD and the tangents at P, P′ concur at the ideal point γ, and are therefore parallel. Q.E.D.

Definition. If the chord HK is a double ordinate to the diameter PP′ and cuts it at N, then HN is called the **ordinate** from H to PP′.

EXERCISE IV. c.

1. Prove by projection that the centre of a conic is inside, on, or outside the curve according as the conic is an ellipse, parabola or hyperbola. What can you say about the asymptotes of the three types of conics?

2. If two chords of a conic bisect each other, prove that each must be a diameter.

3. T is the pole of a chord PQ of a conic, R is the mid point of PQ, prove that TR passes through the centre of the conic.

4. The tangents at two points P, Q are parallel, prove that PQ is a diameter.

5. PP′ is a diameter of a conic, Q is any point on the curve; prove that PQ, QP′ are parallel to a pair of conjugate diameters.

6. PCP′ is a diameter of a conic, centre C; QV is the ordinate from any point Q on the curve to PP′; the tangent at Q meets CP at T; prove that CV . CT = CP²

7. QV is the ordinate from a point Q on a conic to a diameter PP′, a line through P parallel to the tangent at Q cuts QV, QP′ at H, K, prove that PH = HK.

8. PP′ is a diameter of a conic, the tangent at any point R meets the tangent at P in N, P′R meets PN at Q, prove that PN = NQ.

9. The normal QR at a point Q on a conic is an ordinate to the diameter PP′, prove that QR bisects ∠PQP′.

10. QQ′ is a double ordinate to the diameter PP′ of a conic, centre C, R is any other point on the curve, RQ, RQ′ meet PP′ at L, M, prove that CL . CM = CP²

11. A conic touches the sides BC, CA, AB of a triangle at P, Q, R, A′ is the mid point of BC, prove that (i) the pole of AA′ lies on a line through A parallel to BC, (ii) AA′, QR and the diameter through P are concurrent.

12. AB, AC are two chords of a conic, the diameter conjugate to AB meets AC at D, P is the pole of BC, prove that PD is parallel to AB.

13. From a fixed point O is drawn a variable line cutting a given conic at P, Q, PN, QM are the ordinates to the diameter through O, prove that $\frac{1}{ON} + \frac{1}{OM}$ is constant.

14. P, Q are conjugate points w.r.t. a conic σ, if the mid-point of PQ lies on σ, prove that PQ is parallel to an asymptote of σ.

15. PQ is a fixed diameter of a conic σ, D is a fixed point on σ. Two variable chords DH, DK cut PQ at H′, K′, if PH′ = K′Q, prove that the locus of the pole of HK is a straight line.

16. P is any point on an ellipse, PQ, PR are chords cutting the major axis at points equidistant from the centre. The tangents at Q and R intersect in T; prove that PT is bisected by the minor axis.

The Parabola. A conic which touches the line at infinity is a parabola, and the centre of the parabola, defined as the pole of the line at infinity, is a point at infinity, namely the point of contact of the parabola with the line at infinity; further, all diameters of a parabola, defined as lines through the centre, are parallel since the centre is a point at infinity. The proof of Theorem 33 applies equally to the parabola, and we therefore see that mid-points of parallel chords of a parabola lie on a diameter.

Theorem 35. T is the pole of a chord PQ of a parabola; the diameter through T meets the curve at V and PQ at N; then TV = VN and PN = NQ.

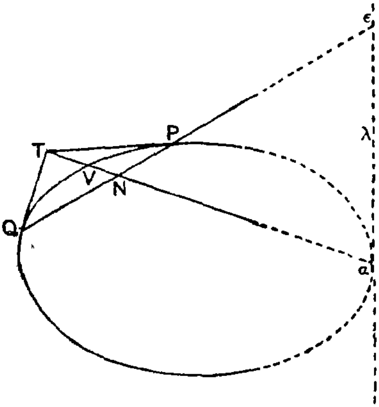

FIG 27

Let the parabola touch the line at infinity λ at the ideal point α, and let PQ meet λ at ϵ.

Since PQ is the polar of T, $\{TVN\alpha\}$ is harmonic;

$$\therefore \; TV = VN.$$

Again, since the polar of T passes through ϵ, the polar of ϵ passes through T; but the polar of ϵ passes through α.

$\therefore$ the polar of ϵ is $T\alpha$, and so $\{PQ\,;\,N\epsilon\}$ is harmonic.

$\therefore$ PN = NQ. Q.E.D.

Note. The locus of mid-points of chords parallel to PQ is the polar of ϵ, the point at infinity on PQ.

EXERCISE IV. d.

1. If PQ is a chord of a parabola, the tangents at P, Q meet on the diameter bisecting PQ

2. Prove that a parabola cannot have two finite parallel tangents

3. Deduce a property of the parabola from Brianchon's theorem · if a hexagon circumscribes a conic, the three lines joining pairs of opposite vertices are concurrent

4. T is the pole of a chord PQ of a conic, R is the mid point of PQ TR meets the curve at V, if $TV = VR$, prove that the conic is a parabola

5. A, B, C are three fixed points on a parabola, P is a variable point on the curve, PB, PC cut the diameter through A at B′, C′, prove that $\dfrac{AB'}{AC'}$ is constant

6. T is the pole of a chord HK of a parabola, any diameter cuts TH, TK, HK at P, Q, R and the curve at V, prove that $VP \cdot VQ = VR^2$. [Let α be the point at infinity on the curve, and note that

$$H\{HVK\alpha\} = K\{HVK\alpha\}]$$

7. T is the pole of a chord PQ of a parabola, any other tangent cuts TP, TQ at H, K, prove that HK is bisected by the tangent parallel to PQ. [Use Chasles' theorem]

8. T is the pole of a chord PQ of a parabola, any diameter cuts TP, TQ at P′, Q′ and a line through T parallel to PQ at R, prove that $P'R = RQ$

9. T is the pole of a chord PQ of a parabola, a line through P parallel to TQ meets the diameter through Q at R, TR meets PQ at H, prove that $PH = 2HQ$ [Bisect PQ at V and prove {PH, VQ} is harmonic]

10. **ABC is a fixed triangle circumscribing a parabola, a variable tangent meets BC, CA, AB at X, Y, Z, prove that $\dfrac{XY}{YZ}$ is constant. [Use Chasles' theorem]**

11. T is the pole of a chord PQ of a parabola, a diameter meets PQ in M, the curve in B and a line through T parallel to the tangent at B in N, prove that $NB = BM$

12. **A variable tangent to a parabola cuts two fixed tangents at X, Y; prove that the locus of the mid-point of XY is a straight line.**

13. PQ is a chord of a parabola, any diameter cuts PQ in N, the curve in B and the tangent at P in R, prove that $\dfrac{RB}{BN} = \dfrac{PN}{NQ}$

14. A parabola is inscribed in the given triangle ABC, touching BC at the given point D, construct its point of contact with AB

15. A chord PQ of a parabola cuts the axis in R, the tangents at P and Q intersect in T, prove that RT is bisected by the tangent at the vertex

The Hyperbola. Since the centre is the pole of the line at infinity the asymptotes are the lines joining the centre to the points of intersection of the curve with the line at infinity. Consequently an ellipse has two imaginary asymptotes, a hyperbola has two real asymptotes, and in the case of a parabola each asymptote coincides with the line at infinity. If the conic is the projection of a circle, the asymptotes are the projections of the tangents to the circle at its points of intersection with the vanishing line.

Theorem 36. Any two conjugate diameters of a hyperbola are harmonically conjugate to the asymptotes.

Conversely, any two lines harmonically conjugate to the asymptotes are conjugate diameters.

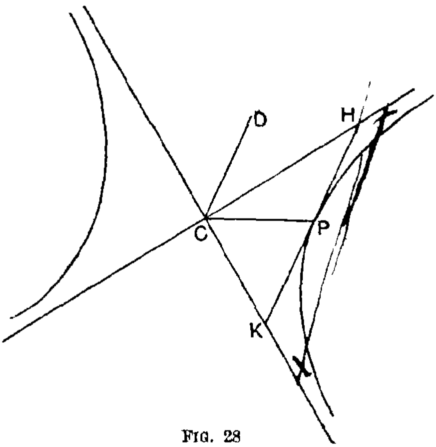

FIG. 28

This is merely a special case of Theorem 27, p. 49.

Theorem 37. If a tangent at any point P of a hyperbola meets the asymptotes at H, K, then HP = PK.

This is merely a special case of Theorem 29, p. 50.

Corollary. If a chord QR of a hyperbola is produced to cut the asymptotes at Q′, R′, then QQ′ = RR′.

If in Fig. 28, HPK is parallel to QR, CP produced bisects QR and Q′R′.

Theorem 38. Conjugate diameters of a rectangular hyperbola are equally inclined to the asymptotes.

By Theorem 36, they are harmonically conjugate to the asymptotes; but the asymptotes are at right angles.

Therefore they are equally inclined to the asymptotes. Q.E.D.

Theorem 39. If a variable tangent to a hyperbola, centre C, cuts the asymptotes at P, P′, then CP . CP′ is constant.

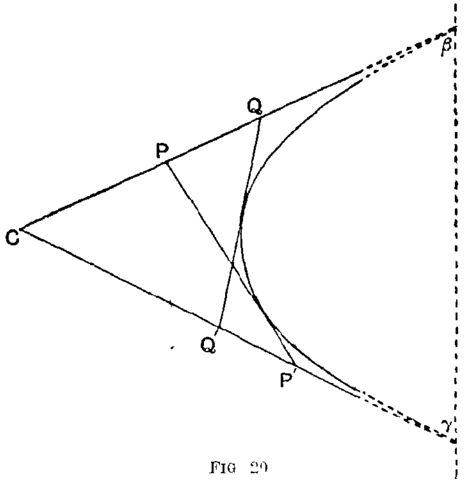

FIG. 29

Let β, γ be the ideal points on the hyperbola, so that $C\beta$, $C\gamma$ are tangents. Draw any other tangent cutting the asymptotes at Q, Q′.

Since by Chasles' theorem PP′, QQ′ and tangents very close to $C\beta$, $C\gamma$ cut off on $C\beta$, $C\gamma$ equicross ranges, we have in the limit

$$\{CP\beta Q\} = \{\gamma P'CQ'\}:$$

$$\frac{CP \cdot \beta Q}{CQ \cdot \beta P} = \frac{\gamma P' \cdot CQ'}{\gamma Q' \cdot CP'}; \quad \text{but } \frac{\beta Q}{\beta P} = 1 = \frac{\gamma P'}{\gamma Q'},$$

$$\therefore \frac{CP}{CQ} = \frac{CQ'}{CP'} \quad \text{or} \quad CP \cdot CP' = CQ \cdot CQ',$$

$$\therefore CP \cdot CP' \text{ is constant.}$$ Q.E.D.

Corollary. A variable tangent to a hyperbola makes with the asymptotes a triangle of constant area.

EXERCISE IV. e.

1. Prove that each asymptote of a hyperbola is a self conjugate diameter.

2. T is the pole of a chord PQ of a hyperbola, a line through T parallel to an asymptote cuts the curve at H and PQ at R, prove that TH = HR.

3. PP′ is a diameter of a rectangular hyperbola; Q is any point on the curve; prove that PQ, P′Q are equally inclined to each asymptote.

4. T is the pole of a chord HK of a hyperbola, TH, TK, HK meet one asymptote at P, Q, R, prove that PR = RQ.

5. A, B, C are three fixed points on a hyperbola, P is a variable point on the curve, PB, PC meet the line through A parallel to one asymptote at B′, C′, prove that $\frac{AB'}{AC'}$ is constant.

6. A line meets the asymptotes of a hyperbola in R, R′ and a pair of conjugate diameters in K, K′, if O is the mid point of RR′, prove that $OR^2 = OK \cdot OK'$.

7. T is the pole of a chord MN of a hyperbola, any straight line parallel to one asymptote cuts TM, TN, MN at P, Q, R and the curve at B, prove that $BP \cdot BQ = BR^2$.

8. P, Q are two points on a hyperbola centre C, lines through P parallel to the asymptotes cut CQ at H, K, prove that $CH \cdot CK = CQ^2$.

9. A, B are two fixed points on a hyperbola, P is a variable point on the curve, PA, PB meet one asymptote at A′, B′, prove that A′B′ is of constant length.

10. PP′ is any diameter of a hyperbola, any chord P′Q meets the lines through P parallel to the asymptote at R, T, prove that RQ = QT.

THE QUADRANGLE AND QUADRILATERAL.

Theorem 40. If a system of conics circumscribe a given quadrangle, the diagonal point triangle is a self-conjugate triangle w.r.t. each conic of the system.

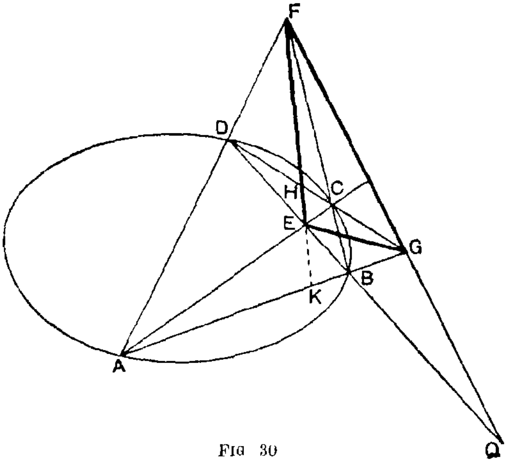

FIG 30

The proof is identical with that employed for the corresponding property of the circle. (See *Modern Geometry*, p. 102.)

By the harmonic theory of a quadrangle, {GCHD} and {GBKA} are harmonic

∴ the polar of G passes through H and K;

∴ the polar of G is HK or EF.

Similarly, the polar of F is EG.

∴ EFG is a self-conjugate triangle

Theorem 41. If a system of conics is inscribed in a given quadrilateral, the diagonal line triangle is a self-conjugate triangle w.r.t. each conic of the system.

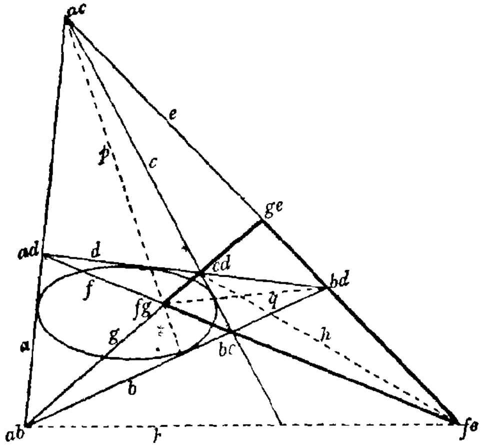

FIG 31

The proof is identical with that employed for the corresponding property of the circle. (See *Modern Geometry*, p. 103.)

Theorem 42. If a quadrangle is inscribed in a conic, and if a quadrilateral is formed by drawing the tangents at its vertices, then the diagonal line triangle of the quadrilateral coincides with the diagonal point triangle of the quadrangle.

The proof is identical with that employed for the corresponding property of the circle.

EXERCISE IV. f.

1. ABCD is a given quadrilateral, a variable conic touches AB, AD at B, D and cuts BC, DC again at H, K, prove that HK cuts BD at a fixed point

2. A variable conic passes through four fixed points; prove that the tangents at these points meet on fixed lines

3. O is any point inside the triangle ABC, AO, BO, CO meet BC, CA AB at D, E, F, prove that DEF is a self conjugate triangle w r t any conic through A, B, C, O

4. Two chords AB, CD of a conic intersect at O, P, Q are the poles of AD, BC; prove that P, O, Q are collinear.

Deduce a special result by taking BC as the line at infinity

5. Given a conic, show how to construct, with the use of a ruler only, the polar of a given point.

6. Given a conic, construct with the use of a ruler only the pole of a given line

7. Two variable chords PQ, RS of a given conic meet at a fixed point O, a fixed line through O cuts PS, QR at L, M, prove that $\frac{1}{OL}+\frac{1}{OM}$ is constant, taking account of the sense of the lines

8. T is the pole of a chord HK of a hyperbola, centre C, TH, TK meet one asymptote at Q, R and the other at Q′, R′, prove that QR′, Q′R, HK are parallel

9. Straight lines are drawn parallel to the asymptotes of a hyperbola through the mid point V of a chord PP′ and meet the curve at Q, Q′, prove that QQ′ is parallel to PP′

10. A, B are two fixed points, PAQ is a variable chord of a given conic, BP, BQ meet the conic again at P′, Q′, prove that P′Q′ passes through a fixed point

11. By taking one side of the circumscribing quadrilateral in Theorem 42 as the line at infinity, prove the following theorem

Through the vertices of a triangle PQR circumscribing a parabola, lines are drawn parallel to the opposite sides forming a triangle P′Q′R′, then P′Q′R′ is a self conjugate triangle w r t the parabola, and the diameters through P′, Q′, R′ meet the curve at its points of contact with QR, RP, PQ

12. If a variable conic passes through a fixed point A and has a fixed self conjugate triangle PQR, prove that it passes through three other fixed points B, C, D and that PQR is the diagonal point triangle of the quadrangle ABCD

What is the dual theorem ?

Theorem 43. A, B, C, D are four fixed points, no three of which are collinear; P is a variable point such that P{ABCD} is constant: then the locus of P is a conic through A, B, C, D.

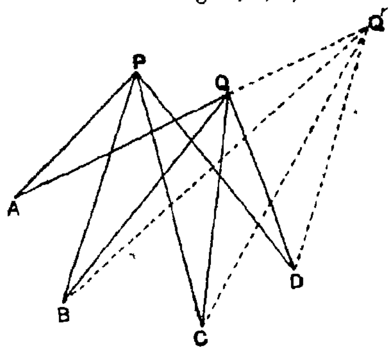

FIG. 32.

Let Q be any other position of P. Draw a conic through P, A, B, C, D [Theorem 24 (3), p. 42], and if it does not pass through Q, let its other meet with AQ be Q′.

Then Q{ABCD} = P{ABCD}, given,
= Q′{ABCD}, Chasles' theorem.

But Q{ABCD} and Q′{ABCD} have a common corresponding ray, therefore B, C, D are collinear, which is contrary to hypothesis.

∴ the locus of Q is the conic through P, A, B, C, D. Q.E.D.

Note. This theorem, which is the converse of Chasles' theorem, may be stated in another form, which is more often useful in rider work:

H, K are two fixed points; HP_1, HP_2, HP_3, ... and KP_1, KP_2, KP_3, ... are two pencils of lines through H, K meeting at P_1, P_2, P_3, ... If the cross ratio of *every four rays* of the first pencil is equal to the cross-ratio of the four corresponding rays of the second pencil, then the points P_1, P_2, P_3, ... lie on a conic through H, K.

To prove the dual theorem, we shall assume that one and only one conic can be drawn to touch five given lines, no three of which are concurrent. This may be established analytically, by using line coordinates.

Theorem 44. *a, b, c, d are four fixed lines, no three of which are concurrent; p is a variable line such that $p\{abcd\}$ is constant; then the envelope of p is a conic touching a, b, c, d.*

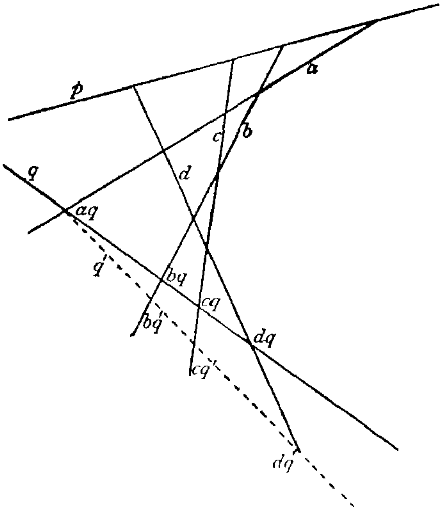

FIG 33

Let q be any other position of p. Draw a conic touching p, a, b, c, d, and if it does not touch q, let the other tangent from aq be q'.

Then $$q\{abcd\}=p\{abcd\}, \quad \text{given},$$
$$=q'\{abcd\}, \quad \text{Chasles' theorem.}$$

But $q\{abcd\}$, $q'\{abcd\}$ have a common corresponding point; therefore b, c, d are concurrent, which is contrary to hypothesis

$\therefore$ the envelope of q is the conic touching p, a, b, c, d. Q.E.D.

Note. This theorem may be stated in another form, which is more often useful in rider work:

h, k are two fixed lines; hp_1, hp_2, hp_3, ... and kp_1, kp_2, kp_3,... are two ranges of points on h, k, whose joins are p_1, p_2, p_3, . . If the

cross ratio of *every four points* of the first range is equal to the cross-ratio of the four corresponding points of the second range, then the lines $p_1, p_2, p_3, \ldots$ envelope a conic touching h, k.

Theorem **44**, *being the dual of Theorem* 43, *is proved in a precisely similar manner, merely making the verbal changes which the Principle of Duality requires, both in the figure and in the argument. In future we shall not write out dual proofs of this kind, as the process is mechanical.*

Theorem 45. A, B are two fixed points; P is a variable point such that PA, PB are conjugate ~~points~~ lines w.r.t. a fixed conic; then the locus of P is a conic through A, B.

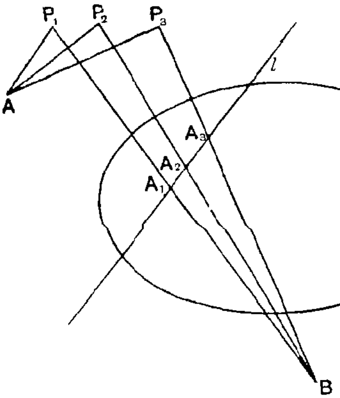

FIG 34

Draw any line AP_1 through A, and let A_1 be its pole.

Join BA_1 and produce it to meet AP_1 at P_1; then AP_1, BP_1 are one pair of conjugate lines; similarly construct any number of other pairs of conjugate lines AP_2, BA_2P_2; AP_3, BA_3P_3; . . .

Now, since the polar of A_1 passes through A, the polar of A passes through A_1; similarly the polar of A passes through A_2, A_3, . . .

$\therefore$ all the points $A_1, A_2, A_3, \ldots$ lie on the polar l of A.

by Theorem 28, p 50, $A\{P_1P_2P_3 \quad\} = \{A_1A_2A_3 \quad\}$

$$= B\{A_1A_2A_3 \quad\}$$

$$\mp B\{P_1P_2P_3 \quad\}$$

by Theorem 43 (the alternative form), the locus of P is a conic through A, B

Corollary 1. The locus of points from which the tangents to a central conic are at right angles is a circle, which is called the **director circle** of the conic

In Theorem 45, take A, B as the circular points at infinity, then the tangents from P to the conic are harmonically conjugate to PA, PB, which are now the isotropic lines through P [Theorem 27]

Therefore the tangents from P to the conic are at right angles [Theorem 1]

But the locus of P is a conic through A, B, and this is a circle since A, B are the circular points at infinity [Theorem 6]

Corollary 2 The locus of points from which the tangents to a parabola are at right angles is a straight line (the directrix)

If, in Theorem 45, we take A, B as the circular points at infinity ω, ω', the line AB or $\omega\omega'$, touches the parabola, and therefore AB is a common corresponding ray of the pencils $A\{P_1P_2, \quad\}$, $B\{P_1, P_2, \quad\}$, and therefore the points P_1, P_2, lie on a straight line Further, if the other tangents from ω, ω' to the parabola touch it at E, F, then E, F are points on the locus, and we shall see later (p 81) that EF is the directrix

The property of the director circle was discovered by De Lahire (1640 1718), who wrote on conics, epicycloids, roulettes and magic squares Its analogy with the directrix of the parabola was pointed out by Boscovich (1711 1787), the name itself is due to Gaskin Some examples on the director circle will be found at the end of Chapter V, as it is better to leave them till after a discussion of the foci

EXERCISE IV. g.

1. If H{ABCD} = K{ABCH}, prove that the conic through H, K, A, B, C touches HD at H.

What is the dual theorem ?

2. The sides AB, AC of the triangle ABC are fixed in position and AB – AC is constant ; prove that BC envelopes a parabola.

3. A is the pole of a fixed chord BC of a given conic ; a variable tangent to the conic cuts AB, AC at L, M ; prove that BM, CL meet on a fixed conic touching AB, AC at B, C.

4. ABC is a given triangle ; P is a variable point on a fixed line BP, CP meet AC, AB at Q, R ; prove that QR touches a fixed conic Does the conic touch AB or BC or CA ?

5. A variable line L cuts the sides of a fixed triangle at X, Y, Z ; if $\frac{XY}{YZ}$ is constant, prove that L envelopes a parabola.

6. The sides AB, AC of a triangle are fixed in position, and BC subtends a given angle at a given point ; find the envelope of BC.

7. A variable tangent to a conic cuts two fixed tangents at P, Q, O is any fixed point ; PP′ is drawn parallel to OQ ; prove that PP′ envelopes a parabola.

8. P is a variable point on the base BC of a fixed triangle ABC ; Q, R are the feet of the perpendiculars from P to AB, AC. What is the *envelope of QR ?*

9. A variable line drawn from a fixed point O cuts the sides AB, AC of the given triangle ABC at P, Q ; find the locus of the meet of BQ, CP

10. A, B are two fixed points ; PQ is a segment of constant length on a fixed line ; prove that AP, BQ meet on a fixed hyperbola, and determine the directions of its asymptotes.

11. The sides QR, RP, PQ of a variable triangle pass through fixed points D, E, F ; P lies on a fixed conic through E, F ; Q lies on a fixed conic through F, D , find the locus of R

12. POQ, PO′Q are two angles of constant magnitudes ; O, O′ are fixed points ; P moves on a fixed line ; prove that Q moves on a conic through O, O′.

13. ABC, PQR are two triangles inscribed in a conic ; prove that their six sides touch a conic.

[Let PQ, PR cut BC at Q′, R′ and AB, AC cut QR at B′, C′, and prove {BQ′R′C} = {B′QRC′}.]

14. P is a variable point on a fixed line L ; A, B are two fixed points ; PQ is a diameter of the circle ABP ; prove that the locus of Q is a hyperbola whose asymptotes are perpendicular to AB and L.

15. A variable line passes through a fixed point; prove that the line joining its poles w r t. two given conics touches a fixed conic inscribed in the common self-conjugate triangle of the two conics.

16. A variable line moves so that its extremities lie on two fixed lines and its mid-point on another fixed line; find its envelope.

17. AP is a variable chord of a given hyperbola. A is a fixed point; a line through A perpendicular to AP cuts a line through P parallel to an asymptote at Q; find the locus of Q.

18. The sides AB, AC of a triangle are fixed in position; the circumcentre of ABC lies on a fixed line; prove that BC envelopes a parabola.

19. Given the base and the difference of the base angles of a triangle, show that the locus of the vertex is a rectangular hyperbola.

20. ABCD is a rhombus; prove that the locus of a point P which moves so that PA, PC are harmonically conjugate to PB, PD is the ellipse whose axes are AC and BD.

21. OBP, OAQ are the asymptotes of a conic, A, B being fixed points and PQ a variable tangent, prove that PA, QB will intersect on a conic which has parallel asymptotes and passes through A, B.

22. A variable line is drawn from a fixed point O to cut two fixed lines AB, AC at P, Q, prove that the locus of the mid-point of PQ is a hyperbola, having its asymptotes parallel to AB, AC.

23. P is a variable point on a fixed line; the polar of P w r t. a given conic meets another given line at Q; find the envelope of PQ.

24. P, Q are conjugate points w r t a given conic; PQ passes through a fixed point and P lies on a fixed line. Find the locus of Q.

What result is obtained by taking the fixed line as the line at infinity?

25. PQR is a self-conjugate triangle w r.t. a given conic, P, Q lie on fixed lines, find the locus of R.

26. (i) Prove that the cross-ratio of the four lines $u=0$, $u-\lambda v=0$, $v=0$, $u-\mu v=0$ is $\dfrac{\lambda}{\mu}$.

(ii) If the equations of the lines HA, HB, HC are $u=0$, $u-\lambda v=0$, $v=0$ and the equations of H′A, H′B, H′C are $u'=0$, $u'-\lambda' v'=0$, $v'=0$, and if HP, H′P are variable lines such that H{ABCP} = H′{ABCP}, prove that the locus of P is the conic $\lambda' u v' - \lambda u' v = 0$, which passes through H, H′.

[*This is an analytical proof of Theorem* 43.]

CHAPTER V

GENERAL PROJECTION

THE contents of this chapter summarise the application to the geometry of the conic of the theory of the circular points at infinity, first developed systematically by Poncelet, in the light of their connection with the foci. The existence of the real foci of a central conic, the name of which is due to Kepler, was known to Apollonius, who showed that the lines joining any point on the curve to the foci are equally inclined to the tangent at that point The earliest mention of the focus of a parabola is found in the writings of Pappus, who showed that the distance of any point on each species of conic from the focus is proportional to its distance from a fixed line (the directrix) It is surprising that this property attracted no notice till Newton drew attention to it in the *Principia* An important advance was made by De Lahire, a pupil of Desargues, who showed that the directrix was the polar of the focus and that conjugate lines through the focus are at right angles

The significance of the circular points at infinity ω ω' has already been explained, they have been defined on an analytical basis, and it is important to remember that their geometrical usage is therefore intelligible only in so far as it connotes an analytical operation For convenience, we shall now enumerate the properties of ω, ω which have already been established

(i) Every circle cuts the line at infinity at two fixed points ω, ω' (p 16)

(ii) Every conic through ω, ω' is a circle [Theorem 6]

(iii) A system of concentric circles have double contact with each other at ω, ω'. [Theorem 4]

(iv) Any pair of lines at right angles are harmonically conjugate to the isotropic lines through their meet; and, conversely, any two lines which are harmonically conjugate to the isotropic lines are at right angles. [Theorem 1.]

(v) If two concurrent pairs of perpendicular lines are each harmonically conjugate to another pair of lines, that pair of lines must be isotropic. [Exercise II. No. 9.]

(vi) If a variable pair of lines include a constant angle, they form with the isotropic lines through their meet a pencil of constant cross ratio, and, conversely, if two variable lines form with the isotropic lines a pencil of constant cross-ratio, then they include a constant angle. [Theorem 1.]

Theorem 46. If C is the centre of a circle, $C\omega$ and $C\omega'$ are the asymptotes of the circle.

C is the pole of the line at infinity, therefore $C\omega$, $C\omega'$ are the tangents to the circle at ω, ω'.

$\therefore$ by definition, $C\omega$, $C\omega'$ are the asymptotes Q.E.D.

Note. This supplies another proof of (iii) above

Theorem 47. If a conic has two pairs of perpendicular conjugate diameters, it must be a circle

Let C be the centre of the conic and P any point on the conic: let PP_1, PP_2, PP_3, PP_4 be the double ordinates to the two pairs of conjugate diameters since these ordinates are perpendicular to the diameters, we have by congruent triangles,

$$CP = CP_1 = CP_2 = CP_3 = CP_4$$

$\therefore$ the circle centre C, radius CP, cuts the conic at five points; but only one conic can be drawn through five points

$\therefore$ the conic must coincide with the circle. Q.E.D.

Note. This theorem may also be proved as follows:

The asymptotes of the conic (*being tangents*) are harmonically conjugate to each pair of conjugate diameters.

$\therefore$ by (v) above, the asymptotes must be isotropic lines.

$\therefore$ the conic passes through ω, ω', and is therefore a circle.

Q.E.D.

Theorem 48. (1) Any conic can be projected into a circle having the projection of any given point (not on the conic) as centre

(2) Any conic can be projected into a circle and at the same time any given line (not touching the conic) into the line at infinity.

(3) Any two points can be projected into ω, ω'

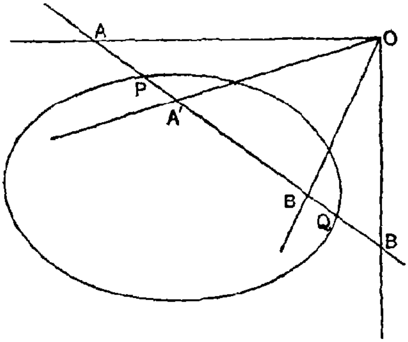

FIG 35

(1) Let O be the given point and AB its polar w r t the conic, draw any two pairs of conjugate lines OA, OA′, OB, OB through O

Project the angles AOA, BOB into right angles and the line AB to infinity [Theorem 21 (2)]

O is therefore projected into the centre of the new conic, and OA, OA′, OB, OB are projected into two pairs of perpendicular conjugate diameters

by Theorem 47, the new conic is a circle, and the projection of O is its centre

(2) Let AB be the given line, then project as in (1) and the required result is obtained.

(3) Let AB be the line passing through the given points P, Q. Draw any conic through P, Q and project the conic into a circle and the line AB to infinity

Then P, Q projects into the meets of a circle with the line infinity, i e ω, ω' Q E D

Theorem 49. (1) If a system of conics have two common points, they can be projected into a system of circles

(2) If a system of conics have four common points, they can be projected into a system of coaxal circles.

(3) If a system of conics have double contact with each other at the same two points, they can be projected into a system of concentric circles.

(4) Any two conics can be projected into two circles

Each of these projections is secured by projecting two common points of the conics into ω, ω

Example. Generalise by projection the following theorem:

If two circles cut orthogonally, the extremities of any diameter of one of the circles are conjugate points w r t the other circle

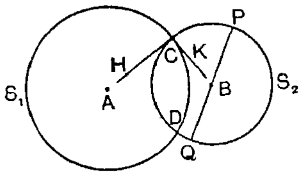

FIG 36

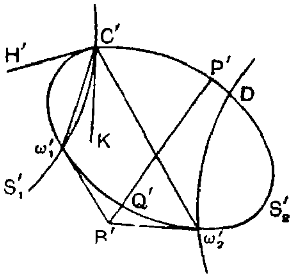

FIG 37

Let A B be the centres of two circles S_1, S_2 which intersect orthogonally at C, D ; let CH, CK be the tangents at C , let PBQ be any diameter of S_2 Denote by dashes corresponding points in the generalised figure

S_1, S_2 intersect at C, D and the circular points at infinity ω_1, ω_2, and may therefore be projected into two conics S_1', S_2' cutting at C , D , ω_1', ω_2 CH, CK project into the tangents C H C K′ to S_2', S_1'

Since $\angle$HCK $=90°$, C{HK , $\omega_1\omega_2$} is harmonic , therefore

$$C'\{H'K , \omega_1 \omega_2\}$$

is also harmonic

B is the pole of $\omega_1\omega_2$ (the line at infinity) w r t S_2, therefore B′ is the pole of $\omega_1'\omega_2'$ w r t S_2' P, Q become the points of inter-

section P′, Q′ of any line through B′ with S_2', and P′, Q′ are conjugate points w.r.t. S_1'.

Hence we have the new theorem (omitting the dashes):

Two conics S_1, S_2 intersect at four points C, D, ω_1, ω_2; if the tangents at C to S_1, S_2 are harmonically conjugate to $C\omega_1$, $C\omega_2$, and if B is the pole of $\omega_1\omega_2$ w.r.t. S_2, then any line through B is cut by S_2 in two points which are conjugate w.r.t. S_1.

Note. In order to acquire facility in applying the method of general projection, it is best first of all to practise the reverse process. The reader is therefore advised to do a fair proportion of the first 18 questions in the following exercise.

EXERCISE V. a.

Generalise by Projection the theorems in Nos. 1-18.

1. If two circles touch at A, and if any line through A meets them again at P, Q, the tangents at P, Q are parallel.

2. If two circles touch each other, the line joining their centres passes through the point of contact.

3. If two circles intersect at A, B, the line joining their centres bisects AB at right angles.

4. If each of three circles touches the other two, the tangents at the points of contact are concurrent.

5. If two circles intersect at A, B, the angle between the tangents at A is equal to the angle between the tangents at B.

6. Angles in the same segment of a circle are equal.

7. The angle in a semi-circle is a right angle.

8. If a variable circle touches two fixed circles at P, Q, then the tangents at P, Q meet on a fixed line, the radical axis of the two circles.

9. If a variable circle touches two fixed circles, the locus of its centre is two conics.

10. If the tangents from a variable point P to a fixed circle include a constant angle, the locus of P is a concentric circle.

11. If PQ, RS are parallel chords of concentric circles, P, Q, R, S are concyclic.

12. ABC is a fixed triangle; if a variable circle passes through A and has B, C as conjugate points, the locus of its centre is a straight line.

13. If the mid point of a variable chord PQ of a circle lies on a fixed line, then PQ envelopes a parabola.

14. If a variable circle passes through a fixed point and touches a fixed line, then the envelope of the polar of another fixed point is a conic.

State also the dual generalised theorem.

15. A line cuts two concentric circles, the first at A, B and the second at C, D ; then AB, CD have the same mid point.

16. If a variable circle touches a given circle and cuts a given line at a given angle, then it touches a second fixed circle, and the line is the radical axis of the two fixed circles

17. A straight line cuts two circles at A_1, A_2 and B_1, B_2, and their radical axis at O, then $\frac{OA_1}{OB_1} = \frac{OB_2}{OA_2}$

18. The angle at the centre of a circle is double the angle at the circumference on the same arc

[Use Laguerre's Theorem, p 7, and note that

$$\cos 2\theta + i \sin 2\theta = (\cos \theta + i \sin \theta)^2 \text{]}$$

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19. If three conics have one common chord, prove that their other common chords are concurrent

20. Two conics have double contact, prove that a chord of one which touches the other is divided harmonically by its point of contact and the common chord.

21. A variable conic passes through two fixed points A, B and touches two fixed lines, prove that the locus of the pole of AB is two straight lines

22. Prove that each pair of common chords of two conics meets a common tangent in points which are harmonic conjugates w r t the points of contact

23. Three fixed conics have double contact with each other at the same two points, prove that any line touching one of them is cut by the others in a constant cross ratio

24. A, B are two fixed points on a fixed conic, P is a variable point on a fixed line, PA, PB meet the conic again at R, S, find the envelope of RS

25. Two triangles are in perspective and their six vertices lie on a conic S ; prove that the pole of the axis of perspective w r t S is the centre of perspective

26. From a fixed point O on a conic, chords OP, OP' are drawn equally inclined to a fixed chord OA ; prove that PP' passes through a fixed point.

[If AB is the chord which subtends a right angle at O, project A, B into ω, ω']

27. A triangle is inscribed in a conic S, two of its sides pass through fixed points A, B, if AB meets S at C, D, prove that the third side envelopes a conic, having double contact with S at C D

28. Two conics have double contact at A, B, prove that the polars of any given point w r t the conics meet on AB

29. ABC is a triangle inscribed in a conic ; prove that the tangents at the vertices meet the opposite sides in three collinear points. [If the points are P, Q, R, project the conic into a circle and PQ to infinity]

30. H, K are the poles of two chords PQ, RS of a conic ; prove that H, K, P, Q, R, S lie on a conic.

31. A variable conic touches two fixed lines at P, Q and passes through two fixed points A, B ; prove that PQ passes through one of two fixed points H, K and that {AB ; HK} is harmonic.

32. If ω, ω' are conjugate points w.r.t. a conic, prove that the conic is a rectangular hyperbola.

33. Three conics pass through four fixed points ; prove that a common tangent to two of them is divided harmonically by the third.

34. Prove by projection the harmonic properties of a conic inscribed in a quadrilateral and circumscribing a quadrangle. [Project the inscribed quadrangle into a parallelogram and the conic into a circle.]

35. PQR is a triangle inscribed in a fixed conic ; PQ, PR are fixed in direction ; prove that QR envelopes a conic having the same asymptotes as the given conic.

36. Prove that the locus of the centres of hyperbolas passing through two fixed points and having their asymptotes parallel to two given lines is a straight line.

37. A is a fixed point ; P is a variable point on a fixed line ; AP cuts a fixed conic at Q, R , find the locus of the harmonic conjugate of P w.r.t. Q, R.

38. T is a pole of a chord PQ of a conic S_1 , a conic S_2 is drawn having TP, TQ as asymptotes ; prove that one pair of common chords of S_1 and S_2 is parallel to PQ.

39. [**Pascal's Theorem.**] **The three pairs of opposite sides of a hexagon inscribed in a conic meet at P, Q, R ; prove that P, Q, R are collinear.** [Project the conic into a circle and PQ to infinity.]

40. A variable conic passes through three fixed points and has two fixed points as conjugate points ; prove that it passes through a fourth fixed point.

Deduce a theorem for a rectangular hyperbola.

41. If a system of conics circumscribing a fixed quadrangle is projected into a system of coaxal circles, prove that two of the diagonal points project into the limiting points of the coaxal system.

42. Two conics touch the same line at P, Q and cut at A, B, C, D ; prove that a conic through A, B, C, D and the mid-point of PQ has one asymptote parallel to PQ.

43. Two parabolas touch at P and cut at Q, R ; prove that PQ, PR are harmonically conjugate to the diameters through P of the two curves.

44. A, B are two fixed points ; P is a variable point such that PA, PB are conjugate lines w.r.t. a fixed conic σ ; prove that the locus of P is a conic through A, B. [Project σ into a circle having the projection of A as centre.]

45. T is the pole of a fixed chord PQ of a conic S ; a variable tangent cuts TP, TQ at H, K ; prove that the locus of the mid-point of HK is a conic, having double contact with S.

THE FOCI.

Definitions.

(1) If a quadrilateral is formed by drawing the pairs of tangents from the circular points at infinity to a conic, and if S, H; S′, H′ are the other two pairs of opposite vertices, then S, H and S′, H′ are called the **foci** of the conic; and the lines SH, S′H′ are called the **principal axes** of the conic. [See Fig. 38, p. 82.]

(2) The polars of the foci w. r. t. the conic are called the **directrices** of the conic; and the chord through a focus parallel to the corresponding directrix is called the **latus rectum.**

(3) If a system of conics have the same foci, they are said to be **confocal.**

Theorem 50. (1) **Every** real conic has four foci, of which two are real and two are conjugate imaginaries.

(2) Every real central conic has two real principal axes, which are a pair of perpendicular conjugate diameters of the conic.

(3) Every pair of conjugate lines through a focus of a conic are at right angles; and, conversely, two perpendicular lines through a focus are conjugate lines.

(1) ω, ω' are conjugate imaginaries; therefore the pair of tangents ωS, ωH are the conjugate imaginaries of the pair of tangents ω'S, ω'H, since the conic is real [see Fig. 38]. But two conjugate imaginary lines meet at a real point.

Therefore either ωS, ω'S and ωH, ω'H *or* ωS, ω'H and ωH, ω'S meet at real points, but not all four pairs, since the lines SS′, S′H, HH′, H′S are imaginary.

Suppose, then, that the first two pairs meet at real points, then S, H are real. In that case ωS, ω'H are the conjugate imaginaries of ωH, ω'S, and therefore their meets H′, S′ are conjugate imaginary points.

(2) Since S, H are real and S′, H′ are conjugate imaginaries, the lines SH, S′H′ are both real.

From the theory of the circumscribed quadrilateral, the mee of SH, S′H′ is the pole of $\omega\omega'$, the line at infinity, and is theref

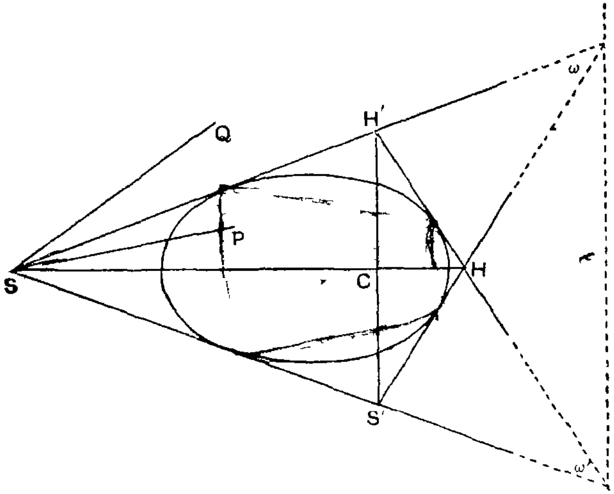

FIG 38

the centre of the conic. Also CH, CH′ are conjugate lines w.r.t. conic; therefore CH, CH′ are conjugate diameters.

But C{HH′; $\omega\omega'$} is harmonic, $\therefore$ $\angle$HCH′ $=90^\circ$.

Therefore the two principal axes are a real pair of perpendic conjugate diameters.

(3) If SP, SQ are a pair of conjugate lines through S, S{PQ; ω is harmonic. [Theorem 27.]

$$\therefore \angle PSQ = 90^\circ.$$

Conversely, if $\angle$PSQ $=90^\circ$, S{PQ; $\omega\omega'$} is harmonic, and th fore SP, SQ are conjugate lines. [Theorem 27.] Q.E.I

Corollary 1. The directrices are parallel to the principal a This follows from the harmonic theory of the circumscril quadrilateral.

Corollary 2. A central conic is symmetrical about each of the two principal axes; a parabola is symmetrical about its (finite) axis.

The parabola needs special consideration.

Since the parabola touches the line at infinity, $\omega\omega'$ at H say, the lines ω'HH$'$, ωHS$'$, H$'$CS$'$ of Fig. 38 all coincide with the line $\omega\omega'$; the points H$'$, S$'$ coincide with ω, ω' respectively, and C coincides with H. A parabola has therefore one focus at a finite point, one focus at its centre which is the point at infinity on the curve, and the other two foci at ω, ω'. It has one finitely situated principal axis, which bisects all chords perpendicular to it, while the other principal axis is the line at infinity. This method of establishing the existence of the foci is simply a statement of an analytical process which can be

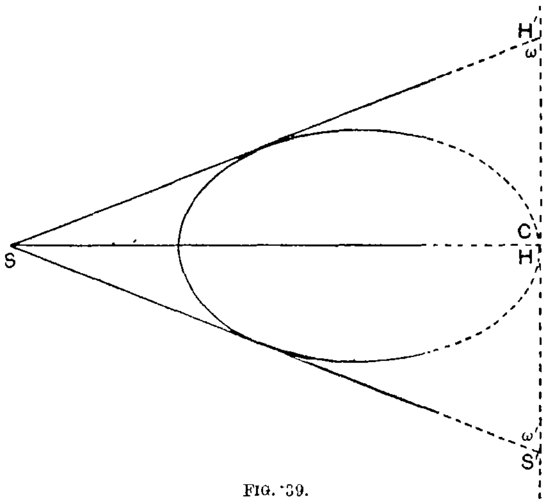

FIG. 39.

expressed more briefly in geometrical language; the figures merely indicate the substance of the arguments employed.

Theorem 51. [Pappus Theorem.] S is the focus of a conic, P is a variable point on the curve, M is the foot of the perpendicular from P to the directrix corresponding to S; then $\frac{SP}{PM}$ is constant.

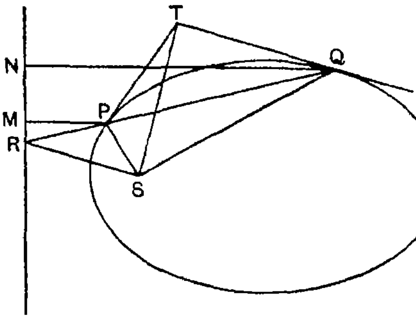

FIG. 40.

Take any other point Q on the curve and draw QN perpendicular to the directrix; produce PQ to meet the directrix at R, and let T be the pole of PQ.

The polar of S, the directrix, passes through R, therefore the polar of R passes through S; also the polar of T passes through R, therefore the polar of R passes through T.

$\therefore$ the polar of R is ST;

$\therefore$ ST, SR are conjugate lines, and so $\angle RST = 90°$.

But S{RPTQ} is harmonic, since ST is the polar of R.

$\therefore$ SR, ST are the bisectors of $\angle PSQ$;

$$\therefore \frac{SP}{SQ}=\frac{PR}{QR}=\frac{PM}{QN} \quad \text{or} \quad \frac{SP}{PM}=\frac{SQ}{QN};$$

$\therefore \frac{SP}{PM}$ is constant. Q.E.D.

Corollary. If the conic is a parabola, $\frac{SP}{PM}=1$.

The principal axis passes through S, cuts the directrix at right angles at X, say, and cuts the curve at A, a, where a is a point at infinity.

Since S is the polar of the directrix, {XASα} is harmonic ;

$$\therefore\ SA = AX \quad \text{or} \quad \frac{SA}{AX} = 1.$$

Definition. The constant value of $\frac{SP}{PM}$ in Theorem 51 is called the **eccentricity** of the conic.

EXERCISE V. b.

1. Prove that the centre is the mid-point of the line joining the foci on either principal axis

2. Prove that the four foci of a circle coincide with its centre Where are the directrices ? Draw a figure, analogous to Fig 38. What is the eccentricity of a circle ?

3. Prove that all parabolas having a common focus may be regarded as inscribed in a common triangle

4. Prove that any conic through the four foci of a conic must be a rectangular hyperbola

5. The centre S of a circle is a focus of a conic Prove that S may be regarded as a meet of two of the common tangents of the circle and conic.

6 S_1, S_2 are two conics inscribed in the quadrilateral ABCD S_3 is any conic through A, B, C, D If S_1, S_2 are projected into confocal conics, prove that S_3 will become either a circle or a rectangular hyperbola

7 A side BC of a triangle ABC, self conjugate w r t a conic, meets a directrix at P , S is the corresponding focus , prove $\angle PSA = 90°$.

8. ABCD is a quadrilateral circumscribing a conic , if the diagonals AC, BD meet at a focus, prove that they are at right angles and that the third diagonal is the directrix

9. Prove that any two conics can be projected into confocal conics

10. If the tangents at two points P, Q of a conic, focus S, meet at T, prove that $\angle$s TSP, TSQ are either equal or supplementary.

11. P is a point on a conic, centre C, focus S , CP meets the corresponding directrix at K , prove that the tangent at P is perpendicular to SK

12. The tangent at P to a conic meets a directrix at R , S is the corresponding focus , prove $\angle PSR = 90°$.

13. The directrix of a hyperbola meets an asymptote CT at T ; S is the corresponding focus , the other tangent from T to the curve touches it at P ; prove that SP is parallel to CT

14. Prove that conics having the same focus and directrix can be projected into concentric circles.

15. Two conics have a common focus and directrix, a chord PQ of one touches the other at H and meets the directrix at K; prove that {PQ; HK} is harmonic.

16. A conic passes through three fixed points and touches a fixed line; prove that the locus of the pole of the line joining two of the points is a conic touching each of the joins of the fixed points

17. Generalise: the circumcircle of a triangle circumscribing a parabola passes through the focus.

18. If a conic and one of its foci are projected into a circle and its centre, prove that any angle at the focus is unaltered in magnitude by projection.

Use this theorem to deduce angle properties of a conic, *e g.* Nos. **10**, 12, 21, 24.

19. Generalise · the orthocentre of a triangle circumscribing a parabola lies on the directrix [Note that the orthocentre is the centre of the circle w.r.t. which the triangle is self-conjugate]

20. Generalise: if a variable circle touches two fixed circles, centres A, B, then the locus of its centre is two conics, each having A, B as foci

21. Two chords PQ, QR of a conic subtend equal angles at a focus, prove that PR meets the tangent at Q on the directrix

22. The tangent at a point P on a hyperbola, focus S, meets an asymptote CT at T, SP meets CT at Q, prove that QT = QS

23. Prove that two ellipses with one common focus cannot intersect in four real points

24. ABC is a triangle circumscribing an ellipse; AB touches the curve at F, S is a focus, prove that the angles ASF, BSC are equal or supplementary

Theorem 52. A system of conics touching four straight lines can be projected into a system of confocal conics.

This will be done if one pair of opposite vertices of the circumscribing quadrilateral are projected into ω, ω'.

Corollary. Any two conics can be projected into two confocal conics.

Theorem 53. A system of conics having a common focus S and a common corresponding directrix L can be projected into a system of concentric circles having the projection of S as centre.

With the usual notation, let Sω, Sω' cut L at P, Q; then each conic touches Sω, Sω' at P, Q

If P, Q are projected into ω, ω', each conic becomes a circle having the projection of S as centre.

Note on Method. For rider work it is important to realise to what extent metrical properties can be transmitted by projection.

(1) Harmonic and cross ratio properties of ranges and pencils are unaltered by projection

(2) If A, B, C are three collinear points, the ratio $\frac{AB}{CB}$ can be transmitted by writing it in the form $\{ABC\infty\}$, where ∞ is the ideal point on AB, in particular, if $AB = BC$, $\{ABC\infty\}$ is harmonic.

(3) The fact that $\angle ABC = 90^\circ$ can be transmitted by writing it in the form $B\{AC, \omega\omega'\}$ is harmonic

The fact that $\angle ABC$ is constant may be re-written as

$$B\{A\omega C\omega'\} = \text{constant}$$

(4) The condition that a conic is a circle is expressed by saying it passes through ω, ω'

(5) The condition that a conic is a parabola is expressed by saying it touches $\omega\omega'$

(6) The condition that a conic is a rectangular hyperbola is expressed by saying that ω ω' are conjugate points w r t it

(7) The condition that S is a focus of a conic Σ is expressed by saying that $S\omega$, $S\omega'$ are tangents to Σ, also the directrix is the polar of S, the centre is the pole of $\omega\omega'$, the asymptotes are the tangents at the points of intersection of the curve and $\omega\omega'$

As has been already noted, it is frequently easier to see how to prove a theorem which is expressed in general rather than in special terms consequently it may be desirable to begin by generalising the given problem and then project it afresh by a different method. Sometimes it is simpler to project the *dual* general theorem, the validity of this method will be established in the chapter on Reciprocation

EXERCISE V. c.

Generalise the properties stated in Nos 1 8

1 P is a variable point on a given circle, C is a fixed point. CP is produced to Q so that $\frac{CP}{CQ}$ is constant then the locus of Q is another circle

2. The locus of the foci of ellipses which touch the four sides of a given parallelogram is a rectangular hyperbola circumscribing the parallelogram

3. The locus of the centre of a conic inscribed in a given quadrilateral is the straight line through the mid points of the three diagonals

4. If a variable circle touches a fixed conic at a fixed point and cuts it again at P, Q, then PQ is fixed in direction

5. If two conics have the same directrix, then their common points are concyclic

6. The director circles of all conics touching four fixed lines are coaxal.

7. The envelope of the polars of a given point w r t a system of confocal conics is a parabola touching the axes of the system

8. AB is a chord of a given circle fixed in direction, a point P divides AB in a given ratio, then the locus of P is a conic having double contact with the circle.

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9 Four conics pass through the points A, B C, D, prove that the cross ratio of the pencil formed by the tangents at A to the four conics is the same as that at B or C or D

10 T is the pole of a fixed chord AB of a given conic Σ, PQ is a variable chord of Σ such that T{PAQB} is constant, show that the envelope of PQ is two conics each having double contact with Σ at A, B

11. A system of conics have double contact at the fixed points A, B; prove that the poles of a given line CD w r t the system lie on another line CE such that C{AB, DE} is harmonic

12 **If two triangles ABC, PQR are such that the sides of one are the polars of the vertices of the other w r t a conic, prove that the triangles are in perspective** [Project the conic into a circle having the projection of A as centre]

13 Two conics cut at A, B, C, D, a variable line through A meets the conics at P, Q, prove that B{PCQD} is constant

14 A variable parabola touches the sides of a fixed triangle ABC, prove that each chord of contact passes through a fixed point

What result is obtained by taking B, C as ω, ω

15 AA, BB, CC′, DD **are four concurrent chords of a conic, if P is any other point on the conic, prove that** P{ABCD} P{A B C D}

16 p, q are two fixed lines touching a given conic Σ, a variable conic S touches p, q and two other fixed lines, prove that the other two common tangents of S, Σ meet on a fixed line [Project the dual theorem.]

17 S_1, S_2, S_3 are three conics inscribed in the same quadrilateral, P is a point of intersection of S_1, S_2, prove that the tangents at P to S_1, S_2 are harmonic conjugates w r t the tangents from P to S_3 [Project the dual theorem]

18 A triangle is circumscribed to a given conic, two of its vertices move on fixed lines which meet at T, prove that the locus of the third vertex is a conic having double contact with the given conic, the chord of contact being the polar of T.

19. **Prove that the asymptotes of all conics which touch two given straight lines at given points envelope a parabola.**

20. A, B are fixed points on two of the sides of a given quadrilateral, the other tangents from A, B to any conic inscribed in the quadrilateral meet at P; prove that the locus of P is a straight line.

21. **If two pairs of opposite vertices of a complete quadrilateral are conjugate points w.r.t. a conic, prove that the third pair are also conjugate points.** [P, Q are conjugate points w.r.t. a circle if the circle on PQ as diameter is orthogonal to the circle.]

22. The asymptotes of two hyperbolas are parallel; prove that their common chord is parallel to one of the diagonals of the parallelogram formed by the asymptotes.

23. Any circle through the centre C of a rectangular hyperbola cuts the curve at H, K; PQR is a triangle self-conjugate w.r.t. the hyperbola; prove that one pair of common chords of the circle and the conic through H, K, P, Q, R are conjugate lines w.r.t. the hyperbola.

24. Two conics have four-point contact at A, any line through A meets the conics at P, Q; prove that the locus of the meet of the tangents at P, Q is a straight line.

The Director Circle.

25. If a, b are the lengths of the semi-axes of a central conic, prove that the radius of the director circle is $\sqrt{(a^2+b^2)}$.

26. What is the director circle of a rectangular hyperbola?

27. Prove that the director circle of a conic passes through the points of intersection of the conic with its directrices.

28. A chord PQ of the director circle, centre C, touches the conic; prove that CP, CQ are conjugate diameters.

29. Prove that the tangent from any point on the director circle of a conic to its auxiliary circle is equal to the semi minor axis.

30. An ellipse of given size slides between two fixed perpendicular lines; find the locus of its centre.

31. **P is a point on the directrix of a conic, focus S; PT is the tangent from P to the director circle; prove that PT = PS.**

32. ABCD is a rectangle circumscribing a conic; AB meets a directrix in P, S is the corresponding focus; prove that $\angle$PSA = $\angle$PBS.

33. **Two conics are such that an unlimited number of quadrilaterals can be inscribed in one and circumscribed to the other. If ABCD is any one such quadrilateral, prove that AC, BD meet at a fixed point.** [Use projection.]

34. Prove that the focus is a limiting point of the coaxal system having the directrix as radical axis and the director circle as a circle of the system.

CHAPTER VI

CELEBRATED PROPERTIES OF THE CONIC

THE initial theorem of this chapter was first enunciated, in complete generality, by Pascal (1623 1662) at the age of sixteen under the name of the Mystic Hexagram He first proved it for the circle and then extended it to the conic, by projection The special case, however, where the conic degenerates to two straight lines is contained in the Porisms of Euclid unproved, and was formally proved by Pappus six centuries later But the credit of Pascal s discovery must be ascribed partly to Desargues in whose work, as we shall see later (p 187), the theorem is implicitly contained The dual theorem was discovered by Brianchon (1806) by an application of polar properties, and is indeed the earliest professedly dual property enunciated. The theorem attached to Pappus name and known as the "Locus ad quatuor lineas" has an interesting record It is mentioned by Pappus as well known, but no proof was given until Descartes solved it in 1659 by means of his analytical methods, a purely geometrical solution was afterwards obtained by Newton and published in the *Principia*

The influence of Desargues and Pascal is evident in the two great works of Carnot, the *Geometry of Position* (1803) and the *Theory of Transversals* (1806) Carnot took a leading part in the revolutionary changes in France at the end of the eighteenth century, and after his banishment in 1796 devoted himself to mechanics and geometry The last theorem of this chapter is associated with the name of Sir Isaac Newton (1642 1727), although it was known to Pappus, and in essence is due to Apollonius. Newton is admittedly the

greatest mathematical genius of all time, yet his researches were almost entirely completed before he was thirty. Owing to his dread of controversy, which publication in those days involved, it was with the greatest difficulty that he could be persuaded to announce his discoveries. The *Principia* was not communicated to the Royal Society until twenty years after it was written. As a geometer, judged by the nature of his methods, he is to be classed among the last of the ancient rather than with the first of the modern school

Theorem 54. [Pascal's Mystic Hexagram]

If a hexagon is inscribed in a conic, then the meets of the three pairs of opposite sides are collinear.

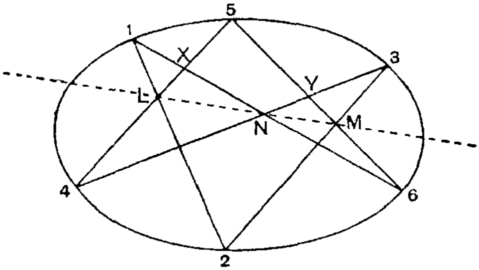

FIG 41

1, 2, 3, 4, 5, 6 are the vertices of the hexagon : 12, 45 ; 23, 56 ; 34, 61 are the pairs of opposite sides intersecting at L, M, N respectively

Let X, Y be the meets of 16, 54 ; 34, 56

Now $\quad 1\{2456\} = 3\{2456\}$, Chasles theorem

But $\quad 1\{2456\} = \{L45X\}$ and $3\{2456\} = \{MY56\}$,

$$\therefore \{L45X\} = \{MY56\}$$

But 5 is a common corresponding point, therefore LM, 4Y, X6 are concurrent But 4Y, X6 meet at N,

∴ L, M, N are collinear Q E.D.

Note. In order to remember this proof, the reader should observe that we start with two pencils whose vertices are two vertices of the hexagon, separated by *one* vertex.

Pascal's Theorem. (Second Proof.)

With the notation of Fig. 41, project LM to infinity and the conic into a circle.

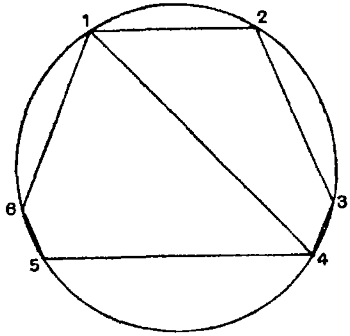

FIG 42

In the projected figure 12, 45 and 23, 56 are parallel ,

$\therefore \angle 143 = 180° - \angle 123 = 180° - \angle 456 = \angle 416$,

$\therefore$ 16 is parallel to 43.

$\therefore$ the meets of 12, 45 , 23, 56 , 34, 61 lie on the line at infinity.

$\therefore$ in the original figure L, M, N are collinear Q E D

Theorem 55. [Brianchon's Theorem]

If a hexagon is circumscribed about a conic, then the joins of the three pairs of opposite vertices are concurrent.

Let ABCDEF be the circumscribing hexagon (Fig 43), then the polars *a*, *b*, *c*, *d*, *e*, *f* of its vertices form an inscribed hexagon Since *a*, *d* are the polars of AD, the point *ad* is the pole of AD. Similarly the points *be*, *cf* are the poles of BE, CF.

But by Pascal's theorem *ad*, *be*, *cf* are collinear.

$\therefore$ their polars AD, BE, CF are concurrent Q E.D.

Note. If we write out the dual method of the first proof given of Pascal's theorem, we obtain a *direct* proof of Brianchon's theorem the reader should do this as an exercise.

It may also be proved by projection as follows :

Let the sides AB, BC, CD, DE, EF, FA of the circumscribing hexagon touch the conic at M, N, P, Q, R, S respectively ; let AD

cut **BE at X.** **Project the conic into a circle having** x, **the projection**

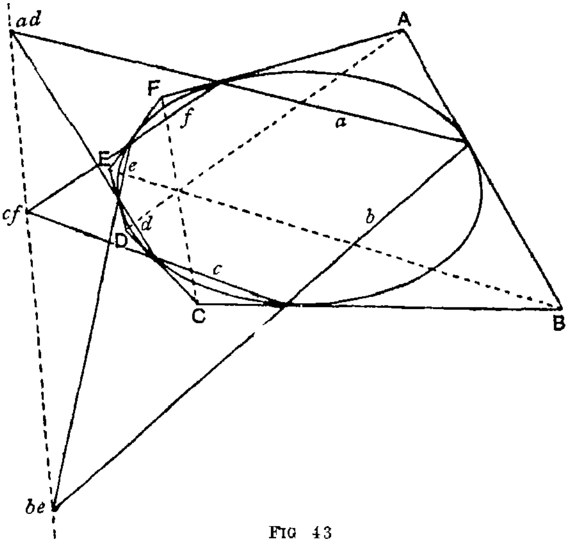

FIG 43

of X, as centre. Use small letters for projections. Since tangents to a circle subtend equal angles at the centre,

$$\angle rxp = 2\angle exd = 2\angle axb = \angle sxn$$

and $$\angle fxr = \angle fxs \quad \text{and} \quad \angle pxc = \angle nxc\,,$$

$$\therefore \ \angle fxr + \angle rxp + \angle pxc = \angle fxs + \angle sxn + \angle nxc\,;$$

$\therefore$ each is 180°, and so fxc is a straight line,

$\therefore$ ad, be, cf are concurrent.

Theorem 56. (1) If the meets of the three pairs of opposite sides of a hexagon are collinear, then its six vertices lie on a conic.

(2) If the joins of the three pairs of opposite vertices of a hexagon are concurrent, then its six sides touch a conic.

These theorems may be proved *either* by reversing the order of the argument in the first proof given of Pascal's theorem and its dual, *or* by an obvious *reductio ad absurdum* method, using the fact that one and only one conic can be drawn to pass through five points or to touch five lines

Theorem 57. (1) If A, C, E ; B, D, F are two sets of three collinear points, then the meets of AD, BC ; CF, ED ; AF, EB are collinear.

(2) If *a, c, e* ; *b, d, f* are two sets of three concurrent lines, then the joins of *ad, bc* ; *cf, ed* ; *af, eb* are concurrent.

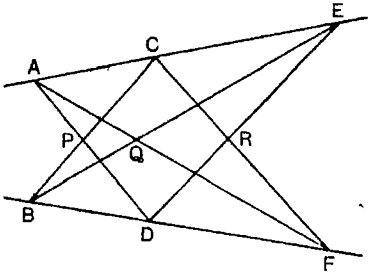

FIG. 44.

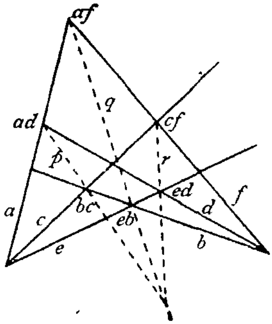

FIG. 45.

These are special cases of Pascal's and Brianchon's theorems when the conic degenerates to (1) two straight lines, (2) two points.

Theorem 58. If a quadrangle is inscribed in a conic, the tangents at its vertices meet in pairs on the sides of the diagonal point triangle.

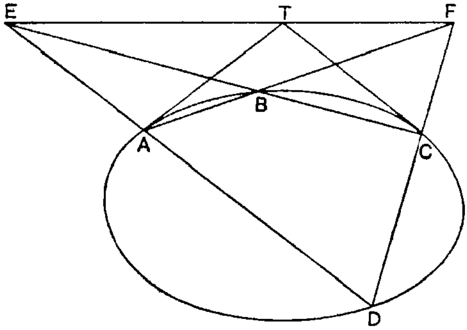

FIG. 46.

Apply Pascal's theorem to the hexagon AABCCD, and it follows that the tangents at A, C meet on EF.

Theorem 59. If a pentagon *abcef* circumscribes a conic, and if *ce* touches the conic at *d*, then *ad*, *be*, *cf* are concurrent.

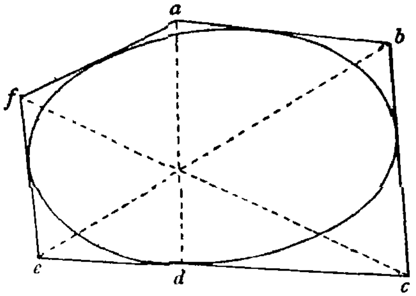

FIG. 47.

This follows at once if Brianchon's theorem is applied to the hexagon *abcdef*.

EXERCISE VI. a.

1. If ABCDEF is a hexagon inscribed in a conic, prove that the meets of AC, DF; DE, BC, AE, BF are collinear.

2. Prove that there are 60 Pascal lines associated with any six points on a conic, and that they form sets of four concurrent lines. What is the dual theorem?

3. Deduce a result from Pascal's theorem by making two consecutive vertices of the hexagon coincide.

4. Deduce a result from Pascal's theorem by making consecutive vertices coincide in pairs

5. Deduce from Brianchon's theorem a property of a triangle circumscribing a conic.

6. Four points P, Q, R, S are taken on a conic; PR meets QS at L; M is any point; PM, QM meet the conic at T, U; prove that ST, RU, LM are concurrent.

7. Deduce, from Brianchon's theorem, a property of a pentagon circumscribing a parabola

8. A variable conic passes through four fixed points A, B, C, D and cuts two fixed lines AP, AQ at P, Q: if CP meets BQ at N, prove that the locus of N is a straight line.

9. T is the pole of a chord PQ of a conic, PH, QK are chords parallel to TQ, TP, prove that PQ is parallel to HK

10 P, Q, R are three points on a hyperbola, QR meets the line through P parallel to one asymptote at S, PQ meets the line through R parallel to the other asymptote at T, prove that ST is parallel to the tangent at Q.

11 P, Q, R, S are four points on a parabola, the diameters through Q, R meet PR, QS at H, K, prove that HK is parallel to PS

12. P, Q, R, T are four points on a conic, QR, PT meet the tangents at T, R in C, D, prove that CD, PQ, RT are concurrent

Deduce a special result by taking RT as the line at infinity

13 B is a fixed point on a parabola, diameters are drawn through the extremities P, Q of a variable chord to meet BQ, BP at L, M, prove that LM is fixed in direction

14 A parabola is inscribed in a triangle ABC touching BC at D, the parallelogram BACN is drawn, prove that DN is a diameter

15. A conic inscribed in the triangle ABC touches AB, AC at H, K, any other tangent to the conic cuts BC, AC at P, Q, prove that BK, QH, AP are concurrent

16. T is the pole of a chord PQ of a parabola R is any other point on the curve, the diameters through P, Q meet QR, PR at H, K, prove that T is the mid point of HK

17. A, B, C are three points on a hyperbola, BC meets one asymptote at D, a straight line AE is drawn parallel to this asymptote to cut a line through D parallel to AB at E Prove that CE is parallel to the other asymptote

18. PQ is a chord of a parabola, through a point T on the tangent at P a line is drawn parallel to PQ meeting the diameter through P at H HQ meets the curve again at R, prove that TR is a diameter

19. If a rectangular hyperbola circumscribes a triangle ABC prove that it passes through the orthocentre [Let α, β be the points at infinity on the curve, let the perpendicular from A to BC cut the curve at H, apply Pascal to BCH$\alpha\beta$A]

20. From any point on the circumcircle of a triangle ABC perpendiculars are drawn to BC, CA, AB and meet the circle again at D, E, F, prove that a conic can be drawn to touch AB, BC, CD, DE, EF FA

21. A hexagon circumscribes a conic S_1 and is inscribed in a conic S_2, prove that its Pascal line w r t S_2 and its Brianchon point w r t S_1 form a side and vertex of the common self-conjugate of S_1 and S_2

22 If a conic touches AB, BC, CD, DA at P, Q, R, S, prove that AC, BD, PR, QS are concurrent.

23. The tangents at two points P, Q on a hyperbola cut one asymptote at H, K and the other at H , K , prove that HK , H K, PQ are parallel

24. T is the pole of a chord PQ of a hyperbola, TP, TQ meet one asymptote at H, K, KP cuts QH at N, prove that TN is parallel to HK.

25. If two triangles are in perspective, prove that the six straight lines joining the vertices of one to the non-corresponding vertices of the other touch a conic.

Constructions. A number of important (first degree) constructions can be effected by means of Pascal's and Brianchon's theorems, the underlying principle is to build up one or other of the fundamental figures of these theorems.

Example I. Given five points on a conic, to construct any number of other points on the conic.

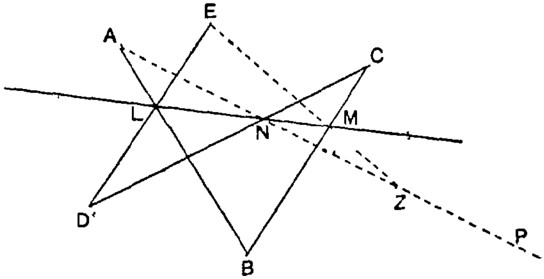

FIG 48

Let ABCDE be the given points, draw through A any line AP, we shall construct the other meet of AP with the conic

Let AB, DE meet at L and CD, AP at N, join LN, and produce it to cut BC at M, join EM, and produce it to cut AP at Z

By construction, the meets of the opposite sides of the hexagon ABCDEZ are collinear, therefore A, B, C, D, E, Z lie on a conic.

Z is the meet of AP with the conic through A, B, C, D, E

Example II. Given five points on a conic, to construct the tangent to the conic at any one of them

Let ABCDE be the given points. [We shall construct the Pascal line of the hexagon AABCDE.]

Let AB, DE meet at L and BC, EA at M, join LM, and produce it to cut DC at N, join NA

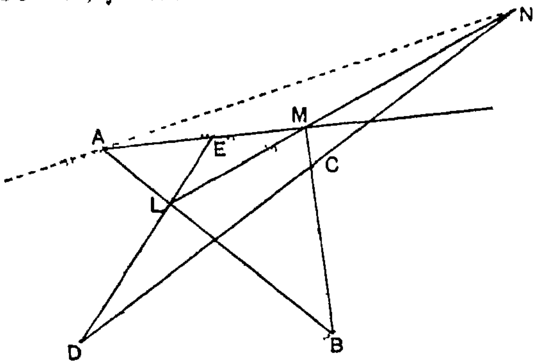

FIG 49

Then LMN is the Pascal line of the hexagon AABCDE; therefore AN is the tangent at A

EXERCISE VI. b.

1. Given five tangents to a conic, construct the point of contact of any one of them [Use Theorem 59]

2 Given four points on a conic and the tangent at one of them, construct any number of points on the curve [Example I, when A coincides with B]

3 Given three points on a hyperbola and one asymptote, construct any number of points on the curve [A special case of No 2]

4 Given four points on a hyperbola and the direction of one asymptote, construct another point on the curve
Construct also that asymptote

5 Given five points on a conic, construct its centre

6 Given five points on a conic, construct the polar of another given point

7. Given four tangents to a parabola, construct their points of contact

8 Given three points on a conic and the tangents at two of them, construct the tangent at the third point

9 Given four tangents to a conic and the point of contact of one of them, construct the points of contact of the others

10. Given the direction of the axis of a parabola and a triangle circumscribing the parabola, find the points of contact with the sides of the triangle

Theorem 60. [Pappus' Theorem.]

A, B, C, D are four fixed points on a given conic; P is a variable point on the conic; PH, PK, PX, PY are the perpendiculars from P to AB, CD, AD, BC; then $\dfrac{PH \cdot PK}{PX \cdot PY}$ is constant.

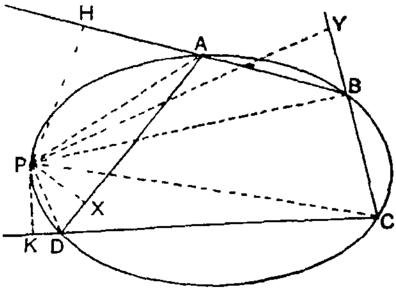

FIG. 50

$$PH \cdot AB = 2\triangle APB = PA \cdot PB \sin APB;$$

similarly

$$PK \cdot CD = PC \cdot PD \sin CPD,$$

$$PY \cdot CB = PC \cdot PB \sin CPB,$$

$$PX \cdot AD = PA \cdot PD \sin APD.$$

$$\frac{PH \cdot PK}{PX \cdot PY} \cdot \frac{AB \cdot CD}{AD \cdot CB} = \frac{\sin APB \cdot \sin CPD}{\sin APD \cdot \sin CPB} = P\{ABCD\}.$$

But $P\{ABCD\}$ is constant, $\therefore \dfrac{PH \cdot PK}{PX \cdot PY}$ is constant. Q.E.D.

Corollary. A, B, C, D are four fixed points; from a variable point P, perpendiculars PH, PK, PX, PY are drawn to AB, CD, AD, BC; if $\dfrac{PH \cdot PK}{PX \cdot PY}$ is constant, the locus of P is a conic through A, B, C, D.

The proof consists merely in reversing the order of the argument given above.

The analytical proof is obvious.

Theorem 61. AB, BC, CD, DA are four fixed tangents to a given conic; p is a variable tangent; AH, BX, CK, DY are the perpendiculars from A, B, C, D to p; then $\frac{AH \cdot CK}{BX \cdot DY}$ is constant.

Let p cut AB at Q, DC at R.

Then, by similar triangles, $\frac{AH}{BX}=\frac{AQ}{QB}$ and $\frac{CK}{DY}=\frac{CR}{DR}$;

$$\therefore \frac{AH \cdot CK}{BX \cdot DY}=\frac{AQ \cdot CR}{QB \cdot DR}.$$

Let any other tangent p' cut AB, DC at Q′, R′. By Chasles' theorem,

$$\{AQBQ'\}=\{DRCR'\} \quad \text{or} \quad \frac{AQ \cdot BQ'}{AQ' \cdot BQ}=\frac{DR \cdot CR'}{DR' \cdot CR};$$

$$\therefore \frac{AQ \cdot CR}{QB \cdot DR}=\frac{AQ' \cdot CR'}{Q'B \cdot DR'};$$

$$\therefore \frac{AQ \cdot CR}{QB \cdot DR} \text{ is constant}; \quad \therefore \frac{AH \cdot CK}{BX \cdot DY} \text{ is constant.}$$

Q.E.D.

Corollary. If AH, BX, CK, DY are the perpendiculars from four fixed points A, B, C, D to a variable line p, and if $\frac{AH \cdot CK}{BX \cdot DY}$ is constant, then p envelopes a conic touching AB, BC, CD, DA.

This theorem is due to Chasles.

EXERCISE VI. c.

1. PL, PM, PN are the perpendiculars from a variable point P to the sides of a fixed triangle; find the locus of P if $\frac{PL^2}{PM \cdot PN}$ is constant.

2. a, b, c, d, e, f are the perpendiculars from a variable point P on a conic to consecutive sides of a given hexagon inscribed in the conic; prove that $\frac{a \cdot c \cdot e}{b \cdot d \cdot f}$ is constant.

3. Extend Pappus' theorem to a polygon of $2n$ sides inscribed in a conic.

4. P is a variable point on a hyperbola; PH, PK are the perpendiculars to the asymptotes; prove that PH . PK is constant.

5. Deduce from Pappus' theorem a property of the parabola by taking B, C as coincident points at infinity.

6. AB is a fixed chord of a hyperbola; parallels through A, B to the asymptotes meet at C; PH, PK, PL are the perpendiculars from a variable point P on the curve to CA, CB, AB; prove that $\frac{PH \cdot PK}{PL}$ is constant.

7. What is the value of the constant in Pappus' theorem if the conic is a circle.

8. Deduce a result from No. 2, by supposing that the hexagon takes the form XXYYZZ.

9. Prove that the constant in Theorem 61 is unity, if the conic is a parabola.

10. A is the pole of a chord BC of a conic; a, b, c are the perpendiculars from A, B, C to a variable tangent to the conic; prove that $\frac{a^2}{b \cdot c}$ is constant.

11. The tangent at a fixed point B of a hyperbola meets the asymptotes EC, ED at C, D. b, c, d, e are the lengths of the perpendiculars from B, C, D, E to a variable tangent; prove that $\frac{b \cdot e}{c \cdot d}$ is constant.

12. A conic touches AB, AC at B, C; if ABC is the triangle of reference, prove that in areal coordinates the equation of the conic is $yz = \lambda \cdot x^2$ By taking AB, AC as the isotropic lines, prove that the distance of a point on the conic from the focus is proportional to its distance from the directrix.

Theorem 62. [Carnot's Theorem.]

If the sides BC, CA, AB of a triangle cut a conic at P_1, P_2; Q_1, Q_2; R_1, R_2 respectively, then

$$\frac{AR_1 \cdot AR_2}{AQ_1 \cdot AQ_2} \cdot \frac{BP_1 \cdot BP_2}{BR_1 \cdot BR_2} \cdot \frac{CQ_1 \cdot CQ_2}{CP_1 \cdot CP_2} = 1.$$

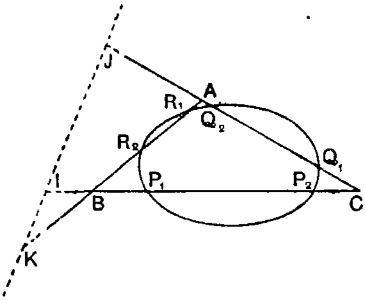

FIG 51.

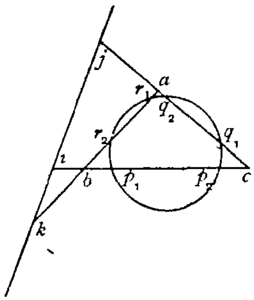

FIG 52

Denote by I, J, K the points at infinity on BC, CA, AB. Project the conic into a circle, and use small letters for corresponding projections. Then i, j, k are three collinear points.

Now $$\frac{BP_1}{CP_1} = \{BP_1CI\}, \text{ since I is at infinity,}$$
$$= \{bp_1ci\} = \frac{bp_1}{cp_1} \cdot \frac{ci}{bi}.$$

Similarly,

$$\frac{BP_2}{CP_2} = \frac{bp_2}{cp_2} \cdot \frac{ci}{bi}; \quad \therefore \frac{BP_1 \cdot BP_2}{CP_1 \cdot CP_2} = \frac{bp_1 \cdot bp_2}{cp_1 \cdot cp_2} \cdot \frac{ci^2}{bi^2}.$$

Similarly,

$$\frac{CQ_1 \cdot CQ_2}{AQ_1 \cdot AQ_2} = \frac{cq_1 \cdot cq_2}{aq_1 \cdot aq_2} \cdot \frac{aj^2}{cj^2} \quad \text{and} \quad \frac{AR_1 \cdot AR_2}{BR_1 \cdot BR_2} = \frac{ar_1 \cdot ar_2}{br_1 \cdot br_2} \cdot \frac{bk^2}{ak^2}.$$

But $bp_1 \cdot bp_2 = br_1 \cdot br_2$, etc.

$$\therefore \text{ the given expression} = \frac{ci^2}{bi^2} \cdot \frac{aj^2}{cj^2} \cdot \frac{bk^2}{ak^2}$$

$= 1$, by Menelaus' theorem, since i, j, k are collinear. Q.E.D.

Corollary. If P_1, P_2; Q_1, Q_2; R_1, R_2 are points on the sides BC, CA, AB of a triangle such that

$$\frac{AR_1 \cdot AR_2}{AQ_1 \cdot AQ_2} \cdot \frac{BP_1 \cdot BP_2}{BR_1 \cdot BR_2} \cdot \frac{CQ_1 \cdot CQ_2}{CP_1 \cdot CP_2} = 1,$$

then these six points lie on a conic.

This is easily proved by a *reductio ad absurdum* method.

Theorem 63. [Newton's Theorem.]

If POQ, ROS are two variable chords of a conic, fixed in direction, then $\dfrac{OP \cdot OQ}{OR \cdot OS}$ is constant.

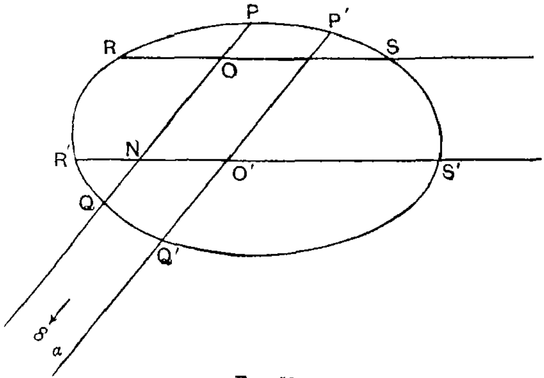

FIG. 53.

Take any other point O′ and draw through it the chords P′O′Q′. R′O′S′ in the given directions; let R′S′ cut PQ at N.

Suppose that α is the point at infinity on PQ, P′Q′, then

$$\frac{\alpha Q}{\alpha Q'} = 1 = \frac{\alpha P}{\alpha P'}.$$

Apply Carnot's theorem to the triangle $\alpha NO'$;

$$\frac{NP \cdot NQ}{NR' \cdot NS'} \cdot \frac{\alpha Q' \cdot \alpha P'}{\alpha Q \cdot \alpha P} \cdot \frac{O'R' \cdot O'S'}{O'P' \cdot O'Q'} = 1;$$

$$\therefore \frac{O'P' \cdot O'Q'}{O'R' \cdot O'S'} = \frac{NP \cdot NQ}{NR' \cdot NS'}.$$

Similarly, $\dfrac{OP \cdot OQ}{OR \cdot OS} = \dfrac{NP \cdot NQ}{NR' \cdot NS'}$; $\quad \therefore \dfrac{OP \cdot OQ}{OR \cdot OS} = \dfrac{O'P' \cdot O'Q'}{O'R' \cdot O'S'}$.

Q.E.D.

Corollary. If CH, CK are the semi-diameters parallel to POQ, ROS, then $\frac{OP \cdot OQ}{OR \cdot OS} = \frac{CH^2}{CK^2}$.

Note. If Newton's theorem is proved independently of Carnot's theorem, it is then easy to deduce Carnot's theorem from it. If in Fig. 51 we draw the semi-diameters OX, OY, OZ parallel to BC, CA, AB, we have

$$\frac{AR_1 \cdot AR_2}{AQ_1 \cdot AQ_2} = \frac{OZ^2}{OY^2}, \text{ etc.},$$

and this gives at once the Carnot relation.

STANDARD EQUATIONS OF THE CONIC

(1) *The Ellipse and Hyperbola.* (Centre at a finite point.)

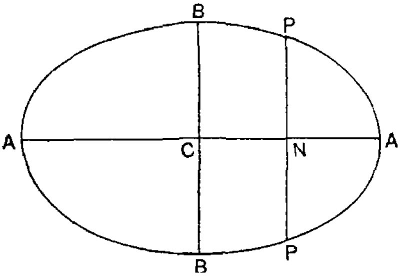

FIG 54

Let ACA′, BCB′ be the pair of perpendicular conjugate diameters: draw any double ordinate PNP′ to AA′.

By Theorem 63, Corollary, $\frac{PN^2}{AN \cdot NA'} = \frac{CB^2}{CA^2}$

Let $CA = a$, $CB = b$, $CN = x$, $NP = y$.

Then $$\frac{y^2}{a^2 - x^2} = \frac{b^2}{a^2} \quad \text{or} \quad a^2y^2 = a^2b^2 - b^2x^2,$$

$$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

This cuts the line at infinity, where $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0.$

If the conic is an ellipse through real points, a^2 and b^2 are each positive.

If the conic is an ellipse with no real point, both a^2 and b^2 are negative.

If the conic is a hyperbola, either a^2 or b^2 is negative, but not both of them; and either the pair of points A, A′ or the pair B, B′ are imaginary, but not both pairs.

(2) *The Parabola.* (Centre at infinity.)

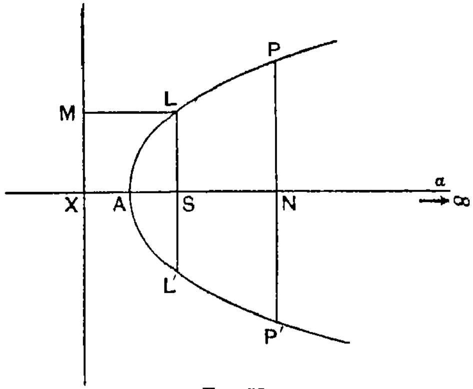

FIG. 55.

Let S be the focus, A the vertex, LSL′ the latus rectum, and α the point at infinity on AS: draw any double ordinate PNP′ to the axis AS.

By Newton's theorem, $\dfrac{\text{NP}\,.\,\text{NP}'}{\text{SL}\,.\,\text{SL}'}=\dfrac{\text{NA}\,.\,\text{N}\alpha}{\text{SA}\,.\,\text{S}\alpha}=\dfrac{\text{NA}}{\text{SA}}.$

Now $\text{SL}=\text{LM}=\text{SX}=2\text{SA}=2a$ (say);

$$\therefore\ \frac{\text{PN}^2}{4a^2}=\frac{\text{AN}}{a}\quad\text{or}\quad \text{PN}^2=4a\,.\,\text{AN}.$$

Let $\text{AN}=x$, $\text{NP}=y$; then $y^2=4ax$.

Note. The fact that *any* **central** conic can be expressed in the form

$$\frac{x^2}{a^2}\pm\frac{y^2}{b^2}=\pm1,$$

enables us to apply the methods of orthogonal projection [Chapter II] to establish properties for any central conic, as defined on p. 5.

In the case of the hyperbola, the transformation, of course, involves imaginary constants. It is not possible to project orthogonally a parabola into a circle, but many properties can be deduced by regarding the parabola as the limit of an ellipse or hyperbola when the centre tends to infinity.

EXERCISE VI. d.

1. What does Carnot's theorem become if the conic degenerates into two coincident straight lines?

2. Extend Carnot's theorem to apply to a quadrilateral.

3. Deduce from Carnot's theorem a result for a conic inscribed in a triangle.

4. Deduce a result by applying Carnot's theorem to the triangle formed by the asymptotes and one tangent of a hyperbola.

5. With the notation of Fig. 51, prove that Carnot's theorem may be written

$$\sin BAP_1 . \sin BAP_2 . \sin CBQ_1 . \sin CBQ_2 . \sin ACR_1 . \sin ACR_2$$
$$= \sin CAP_1 . \sin CAP_2 . \sin ABQ_1 . \sin ABQ_2 . \sin BCR_1 . \sin BCR_2.$$

6. If two conics are inscribed in the same triangle, prove that the six points of contact lie on a conic.

7. A conic cuts the sides BC, CA, AB of a triangle at A_1, A_2, B_1, B_2, C_1, C_2: if AA_1, BB_1, CC_1 are concurrent, prove that AA_2, BB_2, CC_2 are also concurrent.

8. Through any point O lines are drawn parallel to the sides of the triangle ABC; prove that their six meets with the sides lie on a conic.

9. Tangents are drawn from the vertices A, B, C of a triangle to a circle and meet the opposite sides at P_1, P_2; Q_1, Q_2; R_1, R_2. Prove that these six points lie on a conic.

Generalise this theorem and enunciate the dual property.

[Let O be the centre and r the radius of the circle, use No. 5, and note that $\sin BAP_1 . \sin BAP_2 = \sin^2 BAO - \sin^2 P_1AO = \frac{OF^2 - r^2}{OA^2}$, where OF is the perpendicular from O to AB.]

10. PQ is a variable chord of a parabola, fixed in direction; the diameter bisecting PQ cuts the curve at V and PQ at N; prove that $\frac{PN^2}{VN}$ is constant.

11. PQ, HK are two parallel chords of a hyperbola meeting an asymptote at Y, Z respectively; prove that $YP . YQ = ZH . ZK$.

12. Two diameters of a parabola meet the curve at H, K and a chord PQ at H′, K′; prove that $\frac{HH'}{KK'} = \frac{PH' . H'Q}{PK' . K'Q}$.

13. PCP′, DCD′ are two conjugate diameters of a conic; QQ′ is a chord parallel to DD′ cutting PP′ at N, a line through N parallel to PD cuts CD at R, prove that $QN^2 = DR \cdot RD'$

14. T is the pole of a chord PQ of a conic, a chord HK parallel to TP meets PQ, TQ at V, R, prove that $RV^2 = RH \cdot RK$

15. P, Q, R are three points on a parabola, the diameters through Q, R meet the tangent at P in H, K; prove that $\dfrac{HP^2}{PK^2} = \dfrac{HQ}{KR}$.

16 If a conic and a circle intersect at P, Q, R, S, prove that the diameters of the conic parallel to PQ, RS are equal and that PQ, RS are equally inclined to either principal axis.

17. PQ, PR are two chords of a conic, equally inclined to the tangent at P, CH, CK are the semi diameters parallel to PQ, PR, prove that $\dfrac{PQ}{PR} = \dfrac{CH^2}{CK^2}$

18. T is a point on the tangent at a point P of a parabola, any line through T cuts the parabola at Q, R and the diameter through P in H, prove that $TQ \cdot TR = TH^2$

19. Two conics S_1, S_2 cut at A, B, C D, x_1, y_1 and x_2, y_2 are the lengths of the pairs of diameters of S_1 and S_2 parallel to AB, CD, prove that $x_1 y_2 = x_2 y_1$

20 Two conics S_1 S_2 cut at A, B, C, D, from any point P on AB, two lines PH_1K_1, PH_2K_2 are drawn cutting S_1, S_2 at H_1, K_1 and H_2, K_2 respectively, prove that the points C, D, H_1, K_1, H_2, K_2 lie on a conic

21 A chord PQ of a conic subtends equal angles at the extremities of a chord RS, prove that it subtends equal or supplementary angles at the extremities of any chord parallel to RS, if the meet of PQ, RS lies outside the conic

22. The circle of curvature at a point P of a conic cuts a chord PK of the conic at L, x, y are the lengths of the focal chords parallel to PK and the tangent at P, prove that $\dfrac{PK}{PL} = \dfrac{x}{y}$.

CHAPTER VII

RECIPROCATION

THE Principle of Duality springs from the recognition of the fact that a curve may be regarded both as the path of a moving point and as the envelope of a moving line. The latter less obvious idea appears to be due to De Beaune (1601-1652), a student of the work of Descartes. The "Horologium" of Huygens (1629-1695), the inventor of the watch and the earliest writer on the undulatory theory of light, contains some mention of the properties of the evolute (*i.e.* the envelope of the normals) of a parabola and cycloid; and some optical applications to caustics are due to Tschirnhausen (1631-1708). A systematic treatment of envelopes was given in 1692 by Leibnitz, the joint-inventor with Newton of the Calculus. The advantage of coordinating these two conceptions was first pointed out by Brianchon (1806); while the principle itself was stated very clearly by Gergonne in 1825, to whom the notion of the **class** of a curve is due. Poncelet, however, was mainly responsible for the detailed development of the theory, and further applications to metrical properties were made by Chasles and Salmon. The complementary treatment by means of line coordinates is due to Möbius (1790-1868) and Plücker (1829).

Definitions.

(1) The **degree** of a curve is the number of intersections (real or imaginary) of the curve with any straight line.

(2) The **class** of a curve is the number of tangents (real or imaginary) that can be drawn to the curve from any point.

Note. A conic is a curve of the second degree and of the second class.

(3) A, B, C, ... ; l, m, n, ... are any given system of points and lines in a plane, and Σ is any given conic in the same plane. a, b, c, ... ; L, M, N, ... are the polars and poles of this system w.r.t. the conic Σ. Then the system a, b, c, ... ; L, M, N, ... is called the **reciprocal** of the given system w.r.t. the **base-conic** Σ.

It follows at once from the definition that the first system is also the reciprocal of the second w.r.t. Σ. Moreover, the correspondence between the two systems is (1, 1), for to any point and line of the first system corresponds one and only one line and point of the second system.

Notation. Small letters will be used to denote lines and capital letters to denote points: correspondence in the two systems will be indicated by the use of the same letter, *e.g.* the line l in the first system corresponds to the point L in the second.

To avoid confusion, the original figure and the reciprocal figure will usually be drawn out separately.

Theorem 64. (1) If the reciprocals of A, B meet at P, then P is the reciprocal of AB.

(2) If the reciprocals of a, b lie on p, then p is the reciprocal of ab.

(3) If A, B, C, D, ... lie on l, then a, b, c, d, ... pass through L.

(4) If a, b, c, d, ... pass through L, then A, B, C, D, ... lie on l.

These results are all equivalent to the statements that if the polar of A passes through B, then the polar of B passes through A; and if the pole of a lies on b, then the pole of b lies on a.

Theorem 65. (1) If {ABCD} is harmonic, then $\{abcd\}$ is harmonic, and conversely.

(2) The cross-ratio of {ABCD} equals the cross-ratio of $\{abcd\}$, and conversely.

These results follow at once from the theorem that the cross-ratio of any four collinear points is equal to the cross-ratio of the pencil formed by their polars.

Definition. A moving point P traces out a curve S_1, and p is the polar of P w.r.t. the base-conic Σ; then the curve S_2 enveloped by p is called the **reciprocal** of S_1 w.r.t. Σ.

It is of fundamental importance to *prove* that if the curve S_2 is the reciprocal of the curve S_1, then S_1 is also the reciprocal of S_2, as defined above: that is to say, if we take a moving tangent t of S_1, then the locus of its pole T w.r.t. Σ is the same curve S_2 as is obtained by taking a moving point P of S_1 and forming the envelope of the polar p of P w.r.t. Σ. In other words, we must show that we obtain the same reciprocal of S_1 whether we regard S_1 as traced out by a moving point or as the envelope of a moving line.

Theorem 66. If the curve S_2 is the reciprocal of S_1 w.r.t. Σ, then S_1 is the reciprocal of S_2 w.r.t. Σ.

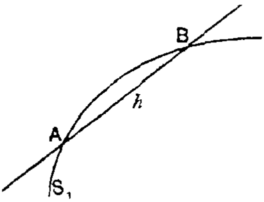

FIG. 56.

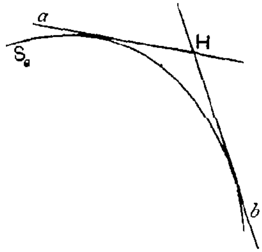

FIG. 57.

Let h be the join of two adjacent points A, B on S_1; the polars of A, B w.r.t. Σ are a, b; therefore the meet of a, b is the pole of AB, and we call it H because it is the reciprocal of h. In the limit, as B tends towards A along the curve, h becomes a tangent to S_1 and H becomes a point on S_2, so that the reciprocal of a tangent to S_1 is a point on S_2.

Therefore S_2 can be generated from S_1 by regarding S_1 as an envelope. Q.E.D.

Theorem 67. The reciprocal of a conic S_1 w.r.t. a base-conic Σ is another conic S_2.

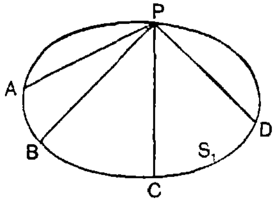

FIG. 58.

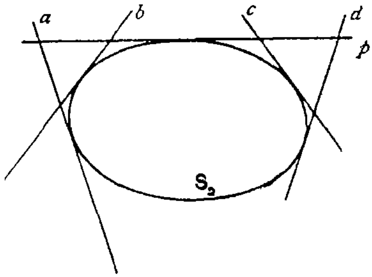

FIG. 59.

Take four fixed points A, B, C, D on S_1, and let P be a variable point on S_1. Then, with the usual notation,

$$p\{abcd\} = \mathrm{P}\{\mathrm{ABCD}\} \quad \text{[Theorem 65 (2)]}$$
$$= \text{constant};$$

$\therefore$ p envelopes a conic, S_2 say, which is the reciprocal of S_1.

Theorem 68. (1) If a line a meets S_1 at P, Q, then the tangents from A to S_2 are p, q; and conversely.

(2) If the join of A, B touches S_1, then the meet of a, b lies on S_2, and conversely.

(3) If A, B, C, D are the common points of S_1, S_1', then a, b, c, d are the common tangents of S_2, S_2', and conversely.

(4) If S_1, S_1' touch at a point A, then a is a common tangent to S_2, S_2' at their point of contact.

All these results follow at once from the fundamental pole and polar properties.

Theorem 69. (1) If a is the polar of B w.r.t. S_1, then A is the pole of b w.r.t. S_2, and conversely.

(2) If P, Q are conjugate points w.r.t. S_1, then p, q are conjugate lines w.r.t. S_2, and conversely.

(3) If PQR is a self-conjugate triangle w.r.t. S_1, then pqr is a self-conjugate triangle w.r.t. S_2.

(1) This follows at once from the harmonic definition of a pole and polar and Theorem 65 (1). (2) and (3) are corollaries of (1).

Theorem 70. (1) If O is the centre of the base-conic Σ, the centre of S_1 reciprocates into the polar of O w.r.t. S_2; and the polar of O w.r.t. S_1 reciprocates into the centre of S_2.

(2) The reciprocal of O is the line at infinity, and conversely.

(3) The asymptotes of S_1 reciprocate into the meets of S_2 witl the polar of O w.r.t. S_2.

(4) S_2 is an ellipse, parabola, or hyperbola according as O lie inside, on, or outside S_1.

These results are all simple special applications of the previou theorems.

Attention has already been called to the fact that all descriptiv theorems occur in pairs; the application of this principle to th geometry of the conic depends on the fundamental property (Theorem 67. The following example illustrates the process in detai

Example. Obtain the dual of the following theorem:

If a variable conic touches four fixed lines, then the locus of th poles of a given line is a fixed line.

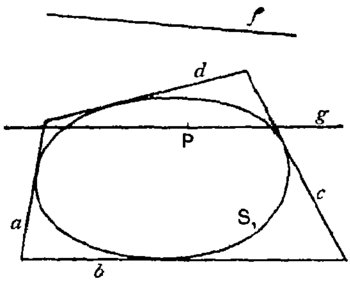

FIG. 60.

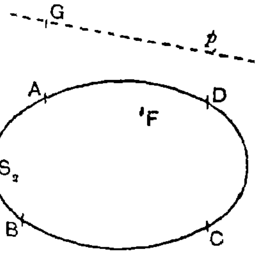

FIG. 61

Let the variable conic S_1 touch four fixed lines a, b, c, d.

Let f be the given line and P the pole of f w.r.t. S_1. Then t reciprocal conic S_2 passes through four fixed points A, B, C, D, a p is the variable polar of the fixed point F w.r.t. the variable conic

Now P traces out a fixed line g; therefore p passes througl fixed point G.

Hence we have the following theorem:

If a variable conic passes through four fixed points, then the polars of a given point pass through a fixed point.

Note. At first, the reader should draw the reciprocal of a given system, bit by bit, and so gradually work up to the dual theorem. But after a little practice, the process can usually be made mechanically, as it involves only certain verbal changes, illustrated in the following two columns. When an element in *either* column occurs, it must be replaced by the corresponding element in the other.

Point	Line
concurrent	collinear
join of two points	meet of two lines
lie on	pass through
quadrangle	quadrilateral
range	pencil
base of range	vertex of pencil
locus	envelope
degree	class
point on a conic	tangent to a conic
meets of a line with a conic	tangents from a point to a conic
pole	polar

EXERCISE VII. a.

1. If two conics touch each other, prove that their reciprocals touch each other.

2. What is the reciprocal of (i) a common chord, (ii) a common tangent of two conics?

3. Two conics cut at P; what is the reciprocal of the tangents at P to the conics?

4. What is the reciprocal of (i) two parallel tangents, (ii) two parallel chords, (iii) a pair of conjugate diameters of a conic?

5. P, Q are two points on a tangent h to a conic S_1; the other tangents from P, Q to S_1 meet at R; construct the reciprocal figure.

6. PQR is a triangle self-conjugate to a conic S_1 and circumscribing a conic S_1'; what is the reciprocal figure?

Write down, without proof, the dual theorems of Nos 7 *to* 21.

7. The sides of a variable triangle pass through fixed non-collinea points, two of the vertices move on fixed lines, then the locus of th third vertex is a conic

If, however, the fixed points are collinear, then the locus of the thir vertex is a straight line

8. If a conic touches the sides BC, CA, AB of a triangle at P, Q, R then AP, BQ, CR are concurrent

9. If a hexagon is inscribed in a conic, the meets of opposite side are collinear

10. A variable triangle is inscribed in a fixed conic, if two of its side pass through fixed points, then the third side touches a conic havin double contact with the given conic

11. Two conics touch at A and cut again at B, C, any line throug A cuts the conics again at P, Q, then the tangents at P, Q mee on BC

12. If three conics have two common points, then the six meets o their common tangents lie three by three on four straight lines

13 If the polars of P, Q, R w r t a conic meet QR, RP, PQ at P′, Q R′ then P′, Q, R′ are collinear

14. A, B are two fixed points, P is a variable point such that th tangents from P to a fixed conic are harmonically conjugate to PA, PB then the locus of P is a conic

15 From a variable point P on a common chord of two fixed conics tangents are drawn meeting the conics at H, K, then HK passes throug one of two fixed points

16. If two triangles are self conjugate w r t a conic, then their si vertices lie on a conic

17. A system of conics have double contact at A, B, C is any othe fixed point, AC, BC cut any one of the conics at P, Q, then PQ passe through a fixed point

18. A variable conic touches two fixed lines and passes through tw fixed points, then the locus of the pole of the join of the fixed point is two lines

19. S_1, S_2, S_3 are three conics inscribed in the same quadrilateral P is a point of intersection of S_1 and S_2, then the tangents at P to S_1, S are harmonically conjugate to the tangents from P to S_3

20. If two conics touch at A and if two lines AP AR cut the conics a P, R and Q, S, then the chords PR, QS meet on the common chord c the two conics

21. From a point on a common chord of two conics, tangents are drawn, one to each conic; then the join of the points of contact passes through a meet of the common tangents.

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22. Reciprocate the Example on p. 112, taking the centre of the base-conic at a corner of the quadrilateral.

23. Reciprocate No. 20, taking the centre of the base-conic at A; deduce a special case when APQ, ARS coincide.

24. Reciprocate w.r.t. a base-conic, centre A: the locus of the centre of a conic touching three fixed lines and passing through a fixed point A is a conic.

25. If the conic S_3 is the reciprocal of the conic S_1 w.r.t. the conic S_2, prove that S_1, S_2, S_3 have a common self-conjugate triangle.

26. Prove by reciprocation and projection: a variable conic is inscribed in a given triangle ABC and has A, D as conjugate points, D being a fixed point; prove that the conic touches another fixed line.

27. Prove by reciprocation and projection, No. 19.

28. Three conics A, B, C touch two given lines; P, Q, R are the points of intersection of the other common tangents of B, C; C, A; A, B respectively: prove that P, Q, R are collinear.

POINT RECIPROCATION

We proceed to consider the special case which arises when the base-conic is a circle. If O is the centre of the base-circle and k its radius, the process is called reciprocating w.r.t. O, and k is called the radius of reciprocation. In general, the value of k is immaterial, for any change in the value of k merely changes the figure into another homothetic figure. The utility of point-reciprocation is due to the fact that the reciprocal of a conic w.r.t. its focus is a circle; two proofs of this will be given, the first depending on an elementary property found in all text-books on geometrical conics, the second following logically from the present method of development.

Theorem 71. The reciprocal of a conic w.r.t. one of its foci S is a circle, and passes through S if the conic is a parabola.

First Method.

Let S be the focus and p a variable tangent to the conic, Y is the foot of the perpendicular from S to p, SY is produced to P so that SY . SP$=k^2$, where k is the radius of reciprocation.

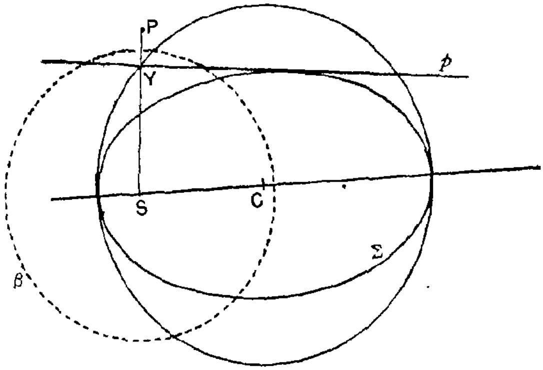

FIG. 62.

Since P is the pole of p w.r.t. the circle, centre S, radius k, P is the reciprocal of p.

Now Y moves on the auxiliary circle of the conic or, if the conic is a parabola, on the tangent at the vertex.

But P traces out the inverse curve of the locus of Y w.r.t. S; therefore the locus of P is a circle, which passes through S if the conic is a parabola. Q.E.D.

Second Method.

Let S be the focus of Σ, and denote the circular points at infinity by ω, ω'; then Sω, Sω' are tangents to Σ.

Now Sω touches the base-circle β, centre S, at ω; therefore the reciprocal of Sω is ω; and so the reciprocal of Σ passes through ω, and similarly through ω'. But the reciprocal of a conic is a conic

(Theorem 67); therefore the reciprocal of Σ is a conic through ω, ω', and is therefore a circle.

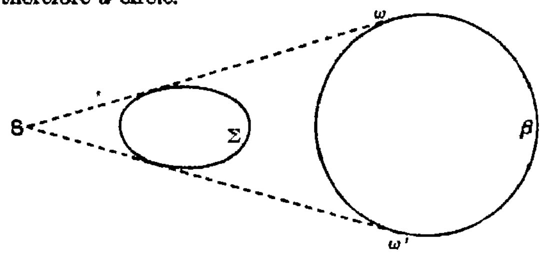

FIG. 63.

Further, if Σ is a parabola, it touches $\omega\omega'$, and therefore its reciprocal passes through S. Q.E.D.

Corollary. The reciprocal of a circle w.r.t. any point S is a conic having one focus at S; and if S lies on the circle, the conic is a parabola.

Notation. For the sake of brevity, a definite notation is adopted for the remainder of the chapter.

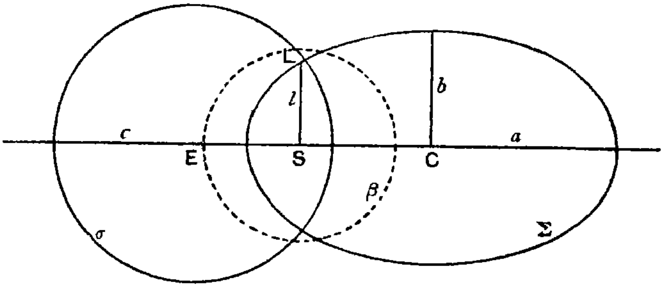

FIG. 64.

The circle σ and the conic Σ are reciprocally related to each other by the base-circle β, whose centre is the focus S of Σ and whose radius is k; E is the centre and c is the radius of σ; C is the centre and a, b are the semi-axes of Σ; l is the length of the semi-latus rectum SL of Σ.

Theorem 72. (1) The point S and the line at infinity are reciprocals.

(2) The points at infinity on Σ are reciprocal to the tangents from S to σ.

(3) The asymptotes of Σ are reciprocal to the points of contact with σ of the tangents from S to σ

(4) Σ is an ellipse, parabola, or hyperbola according as S lies inside, on, or outside σ

These results follow at once from the fundamental properties of poles and polars, and especially the result that the reciprocal of a point and its polar is a line and its pole

Theorem 73. (1) The point C and the polar of S w.r.t. σ are reciprocals.

(2) The directrix of Σ and the point E are reciprocals

(1) C is the pole of the line at infinity w r t Σ, therefore its reciprocal is the polar of S w r t σ [Theorem 69 (1)]

(2) The directrix of Σ is the polar of S w r t Σ, therefore its reciprocal is the pole of the line at infinity w r t σ, that is the centre E of σ. [Theorem 69 (1)] Q E D.

EXERCISE VII. b.

1. Prove that a system of parallel lines reciprocate into a system of points collinear with the origin S

2. What is the reciprocal of the extremities of the minor axis of a conic w r t a focus S?

3. ABC is a triangle, SA meets BC at P, what is the reciprocal of P w r t S?

4. What is the reciprocal of a quadrangle w r t a diagonal point?

5. A circle is reciprocated w r t a point outside it, what are the two parts of the circle which correspond to the two branches of the reciprocal hyperbola?

6 What is the reciprocal of a pair of conjugate diameters w r t a focus of the conic?

7. What is the reciprocal of two circles w r t. a centre of similitude?

8. Reciprocate w r t A two circles touch at A, a line PAQ meets the circles at P, Q; then the tangents at P, Q are parallel.

9. Given four points S, A, B, C, prove that in general four conics can be drawn through A, B, C having S as focus, and that three of them are hyperbolas with A, B, C not all on the same branch, while the remaining one may be an ellipse, parabola or a hyperbola with A, B, C on the same branch.

10. Two conics touch at A, cut at B, C and have a common focus S, if S lies on BC, what is the reciprocal figure w r t S?

11 A conic touches two fixed lines and has a given focus, find the locus of its centre

12. Three conics have a common focus, prove that the meets of the common tangents of the conics, taken in pairs are collinear

13. If two conics have a common focus, prove that a pair of common chords will pass through the meet of the directrices corresponding to that focus

14 A variable conic passes through two fixed points and has a given focus, prove that its directrix passes through one of two fixed points

15 Reciprocate (i) w r t C, (ii) w r t any point the locus of the centre of a variable circle which touches a fixed circle, centre C, and also a fixed straight line is a parabola, focus C

16. A variable conic Σ touches internally each of two fixed conics at variable points P, Q If the three conics have a common focus, prove that the pole of PQ w r t Σ lies on a fixed line

17. Reciprocate (i) w r t A (ii) w r t any point S_1, S_2 are two fixed circles, centres A B, a variable circle Σ touches S_1, S_2, then the locus of its centre is two confocal conics, A, B being the foci.

18 PQ is a double ordinate to the axis of a parabola, if the line joining P to the foot of the perpendicular from the focus S to the directrix cuts the curve again at P′, prove that P Q passes through S

19 A variable parabola touches a fixed conic and has its focus at one of the foci of the given conic, prove that its directrix touches a fixed circle

20. P is a point on a parabola, focus S, SP and the tangent at P meet the directrix in M, M′, the joins of M, M′ to the vertex meet the curve again at Q, Q′, prove that QQ′ passes through S

21 If two of the six meets of four tangents to a parabola lie on the axis, prove that the remaining four are equidistant from the focus

Theorem 74. (1) With the previous notation, $c = \dfrac{k^2}{l}$.

(2) If the perpendicular through S to ES meet σ at H, then

$$b = \frac{k^2}{\mathrm{SH}}.$$

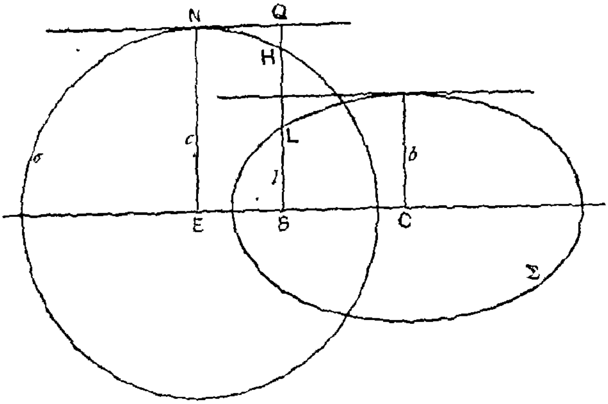

Fig. 65.

(1) Produce SL to Q so that $\mathrm{SL} \cdot \mathrm{SQ} = k^2$; draw QN perpendicular to SQ; QN is by definition a tangent to σ; let N be its point of contact.

Since $\angle \mathrm{ENQ} = 90°$, ESQN is a rectangle.

$$\therefore\ c = \mathrm{EN} = \mathrm{SQ} = \frac{k^2}{\mathrm{SL}} = \frac{k^2}{l}. \qquad \text{Q.E.D.}$$

(2) The reciprocal of H is a tangent to Σ, parallel to the major axis. Therefore the distance of this line from S is equal to the semi-minor axis.

$$\therefore\ b = \frac{k^2}{\mathrm{SH}}. \qquad \text{Q.E.D.}$$

EXERCISE VII. c.

1. *With the usual notation, prove that* $a = \dfrac{k^2}{r^2 \sim \mathrm{ES}^2} \cdot c$.

2. *Prove that the eccentricity of* Σ *equals* $\dfrac{\mathrm{ES}}{c}$.

8. What is the reciprocal of the second focus of Σ ?

4. What is the reciprocal of the minor axis of Σ ?

5. What is the reciprocal of (i) the foot of the perpendicular from S to the corresponding directrix, (ii) two conjugate points on this directrix, (iii) the extremities of the principal axes ?

6. Prove that a system of equal circles reciprocate into a system of conics having a common focus and equal latera recta.

7. Prove that the major axis of an ellipse which is the reciprocal of the circumcircle of a triangle w.r.t. its in-circle is equal to the in-radius.

8. Two coaxal parabolas have a common focus S ; prove that the sum of their latera recta equals 4SP, where P is one of their common points.

9. Two conics with the same focus and directrix are such that triangles can be inscribed in one and at the same time circumscribed to the other ; prove that the eccentricity of one is twice that of the other.

10. Two conics having a common focus intersect at two and only two real points P, Q ; H, K are the poles of PQ w.r.t. the conics ; prove that H, S, K are collinear.

11. I is the in-centre and r the in-radius of an equilateral triangle ABC ; a hyperbola is drawn to circumscribe ABC and have one focus at I ; prove that its eccentricity $= \frac{4}{3}$ and that its latus rectum $= \frac{8}{3}r$.

12. A variable conic has a given focus S and touches two fixed lines OA, OB ; prove that its minor axis envelopes a parabola, of which S and a line through O are focus and directrix.

18. Reciprocate w.r.t. P : T is the pole of a chord QR of a circle PQR ; PL, PM, PN are the perpendiculars from P to QR, TQ, TR ; then $PL^2 = PM . PN$.

14. PQR is a triangle circumscribing a parabola, focus S ; if R′ is the point of contact of PQ, prove that SR . SR′ = SP . SQ.

15. Reciprocate w.r.t. O : the envelope of the polars of a fixed point O w.r.t. a system of equal circles passing through a fixed point is a conic.

16. O is the circumcentre of the triangle ABC : prove that the major axis of the conic inscribed in ABC with one focus at O is equal to OA.

17. PQ is a variable chord of an ellipse, eccentricity e, subtending a right angle at the focus S ; prove that the locus of the pole of PQ is a hyperbola, parabola, or ellipse according as $e >, =, < \frac{1}{2}\sqrt{2}$.

18. H is the orthocentre of a triangle ABC ; S_1 is a conic having H as focus and AB as directrix ; S_2 is a conic having H as focus and AC as directrix. If S_1, S_2 touch BC, prove that their minor axes are equal.

19. A, A′ ; B, B′ ; C, C′ are the pairs of opposite vertices of a quadrilateral circumscribing a parabola, focus S ; prove that

$$SA . SA' = SB . SB' = SC . SC'.$$

20. The sides of a triangle ABC touch a parabola at P, Q, R ; if S is the focus, prove that SA . SB . SC = SP . SQ . SR.

21. A and B are two circles; P is the reciprocal of A w.r.t. B, and Q is the reciprocal of B w.r.t. A; show that the ratio of the latera recta of P and Q is equal to the cube of the ratio of their eccentricities.

22. D, E, F are the mid-points of the sides of a triangle ABC ; O is the orthocentre of the triangle DEF. Prove that the two conics having O as focus and inscribed respectively in the two triangles have equal eccentricities.

Theorem 75. If the lines p, q are the reciprocals of the points P, Q w.r.t. a point S, then the angle PSQ is equal to the angle between p, q.

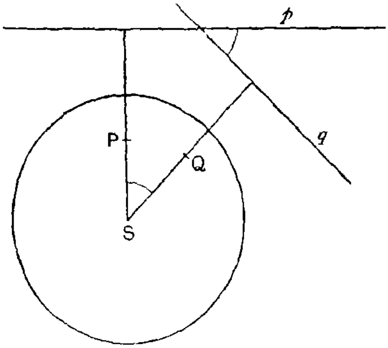

FIG 66

SP, SQ are by definition perpendicular to p, q.

$\therefore$ $\angle$PSQ = angle between p, q. Q.E.D.

EXERCISE VII. d.

1. **What is the reciprocal of :**

(i) **a triangle w.r.t. its orthocentre,**

(ii) **a parabola w.r.t. a point on the directrix,**

(iii) **a conic w.r.t. a point on the director circle ?**

2. What is the reciprocal of two orthogonal circles (i) w.r.t. any point, (ii) w.r.t. a point of intersection ?

3. An asymptote CE of a hyperbola meets a directrix at E ; S is the corresponding focus, prove that $\angle$CES = 90°.

4. Two parabolas have a common focus S, prove that their common tangent subtends at S an angle equal to the angle between their axes.

5. Two chords PR, QR of a conic meet the directrix at L, M; S is the corresponding focus; prove that $\angle LSM = 90° \pm \frac{1}{2}\angle PSQ$.

6. Find the envelope of a variable chord of a conic subtending a constant angle at the focus; find also the locus of its pole.

7. PQR is a variable triangle circumscribing a conic, focus S; $\angle PSQ$ is constant, find the locus of R.

8. If two parabolas have a common focus, prove that their common chord bisects the angle formed by the directrices

9. If two parabolas, having a common focus, cut at P, Q, prove that their common tangent is parallel to a bisector of $\angle PSQ$.

10. If two parabolas have a common focus and their axes in opposite directions, prove that they cut orthogonally.

11. Find the locus of a point from which tangents to a fixed parabola are inclined at a constant angle

12. A conic, inscribed in the triangle ABC, touches BC at A′, S is a focus; prove that $\angle BSA' + \angle ASC = 180°$.

13. O is a fixed point on a conic, PQ is a variable chord such that $\angle POQ = 90°$; prove that PQ passes through a fixed point.

14. Prove that the opposite sides of a quadrilateral circumscribing an ellipse subtend supplementary angles at a focus.

15. If two parabolas have a common focus, prove that the line joining the focus to the meet of the directrices is perpendicular to the common tangent.

16. Prove that the locus of the foot of the perpendicular from the focus of a parabola to a variable tangent is a straight line.

17. ABCD is a quadrilateral circumscribing a conic; AC, BD meet at a focus, prove that AC is perpendicular to BD, and that the directrix is the other diagonal.

18. The tangents from a variable point P to a conic meet the directrix in conjugate points, find the locus of P

19. PQ, PR are two chords of a rectangular hyperbola such that $\angle QPR = 90°$, prove that the tangent at P is perpendicular to QR.

20. The tangent at any point P of a hyperbola cuts an asymptote at T; a line through P parallel to that asymptote cuts the directrix at K, prove that KT subtends a right angle at the corresponding focus.

21. SP is drawn through a focus S of a hyperbola, parallel to an asymptote, and cuts the curve at P; prove that the tangent at P meets the other asymptote on the latus rectum produced.

22. Two conics have a common focus S and two real common tangents; from a variable point P on one of the common tangents, two other tangents PX, PY are drawn to meet the second common tangent in and Y; prove $\angle XSY$ is constant.

23. D is any point on the circumcircle of an equilateral triangle ABC; prove that a parabola can be described having D as focus and touching AB, BC, CA at their meets with DC, DA, DB respectively.

24. On the tangent PT at any point P on a conic, a length PT is measured so as to subtend a right angle at a fixed point inside the conic; prove that the locus of T is the polar reciprocal w.r.t. O of the envelope of normals to the reciprocal conic.

Theorem 76. (1) A system of conics having a common focus can be reciprocated into a system of circles: and if the latera recta of the conics are equal, the circles are of equal radii; and conversely.

(2) A system of conics having a common focus and a common corresponding directrix can be reciprocated into a system of concentric circles; and conversely.

(1) follows at once from Theorems 71, 74 (1).

(2) follows at once from Theorems 71, 73 (2).

Theorem 77. (1) A system of confocal conics can be reciprocated into a system of coaxal circles.

(2) A system of coaxal circles, reciprocated w.r.t. a limiting point becomes a system of confocal conics.

(1) Reciprocate w.r.t. one of the foci: then the conics become circles. But, by definition, a system of confocals have four fixed common tangents. Therefore the reciprocal system of circles have four common points; two of these are the circular points at infinity, and the join of the other two forms the radical axis of the system.

(2) Let L, L′ be the limiting points. Draw L′H perpendicular to LL′; then L′H is the polar of L w.r.t. each circle of the system.

Reciprocate w.r.t. L; then each circle becomes a conic with one focus at L. But L′H, the polar of the origin w.r.t. each circle, reciprocates into the pole of the line at infinity, *i.e.* the centre, of each conic. Therefore the reciprocals are conics having one common focus and a common centre: they are therefore confocal conics.

Q.E.D.

Corollary. When a coaxal system is reciprocated w.r.t. a limiting point, the radical axis reciprocates into the second focus, and the second limiting point reciprocates into the minor axis.

This follows from the fact that the radical axis is mid-way between the origin L and the line L′H, which becomes the centre.

Note. The following alternative treatment of Theorem 77 deserves notice.

S, H ; S′H′ are the pairs of foci of a system of confocals ; Ω, Ω' are the circular points at infinity. Denote the lines SΩ, SΩ',

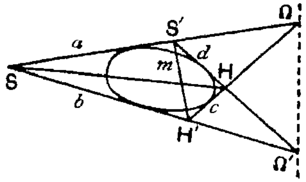

FIG. 67.

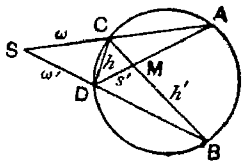

FIG. 68.

HΩ, HΩ' by a, b, c, d. Reciprocate w.r.t. S. The reciprocal of SΩ (or a) is the point Ω, since the line SΩ touches any circle, centre S, at Ω ; we denote this by A, as usual : similarly the reciprocal of SΩ' (or b) is Ω', which we denote by B. Consequently in the reciprocal figure, A, B denote the circular points at infinity, and the lines ω, ω' are the reciprocals of Ω, Ω' ; the lines c, d become points C, D on ω, ω' : the joins of C, D ; C, B ; A, D are h, h', s'.

Any confocal (since it touches a, b, c, d) becomes a conic through A, B, C, D, *i.e.* a circle through C, D. Hence the reciprocal system is a set of coaxal circles having CD or h as radical axis : the radical axis is therefore the reciprocal of the second focus.

Moreover, the point-circles belonging to this system are the isotropic pairs of lines ACS, BDS and AMD, BMC, which yield the point-circle S and the point-circle M, where M is the meet of AD, BC and is the reciprocal of m, the join of ad, bc or S′, H′, *i.e.* the minor axis. The limiting points of the coaxal system are therefore the focus S, taken as origin, and the reciprocal of the minor axis.

Since the figures are reciprocal in every respect, either may be regarded as generating the other, by reciprocation.

EXERCISE VII. e.

1. If two confocals intersect, prove that they cut orthogonally.

2. A system of conics have a common focus and directrix; prove that the polars of any given point w r t the system are concurrent

3 A system of conics have a common focus and touch each of two parallel lines prove that the corresponding directrices are concurrent, that the centres are collinear, and that the asymptotes envelope a circle

4. A line drawn through a limiting point L of a coaxal system of circles cuts one of the circles at A, B The tangents at A, B cut another circle of the system at P, Q and R, S respectively, prove that

$$\angle PLR = \angle QLS$$

5. A system of hyperbolas have a common focus and a common corresponding directrix, find the envelope of the asymptotes

6. Prove that confocal conics of reciprocal eccentricities intersect at the extremities of their latera recta

7. Prove that the locus of the pole of any tangent to the director circle of a conic w r t that conic is another concentric conic

8. Two conics have a common focus S and a common corresponding directrix, a tangent at P to one meets the other at Q, R, prove that $\angle QSP = \angle PSR$

9 SL is the semi latus rectum of a parabola, focus S, an ellipse is drawn through S to have four point contact with the parabola at L, prove that it touches the axis of the parabola

10. A variable circle passes through two fixed points A, B, and cuts two fixed lines through A in P, Q, find the envelope of PQ [Reciprocate w r t B]

11 S is a focus of the conic σ_1, σ_2 is a conic having S as focus and any tangent of σ_1 as the corresponding directrix If σ_2 touches the minor axis of σ_1 prove that the conics are of equal eccentricity

12 The conics S_1, S_2 have a common focus and equal eccentricities if the directrix of S_2 is an asymptote of S_1, prove that the minor axis of S_1 is an asymptote of S_2

Theorem 78. If S_1 and S_2 are two given conics, there exists in general a conic S such that S_1 S_2 are reciprocal w r t S

Project S_1 and S_2 into two conics S_1', S_2', having a common centre and the same principal axes [If PQR is the common self conjugate triangle, this is effected by projecting QR to infinity and $\angle QPR$ into a right angle]

Let $A_1'CA_1$, $B_1'CB_1$ and A_2CA_2, $B_2'CB_2$ be the principal axes of S_1', S_2' Take points A, A , B, B on CA_1, CB_1 such that

$$CA^2 = CA'^2 = CA_1 \cdot CA_2 \quad \text{and} \quad CB^2 = CB'^2 = CB_1 \cdot CB_2.$$

Let S' be the conic having $A'CA$, $B'CB$ as principal axes. Then the reciprocal of S_1' w.r.t. S' passes through A_2, A_2', B_2, B_2' and has A_2A_2', B_2B_2' as principal axes, and therefore coincides with S_2'.

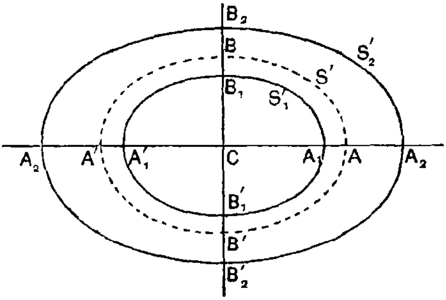

FIG 69.

Therefore S_1' and S_2' are reciprocal w r t S'.

Therefore, projecting back, there exists a conic S w r.t which S_1 and S_2 are reciprocals. Q.E D.

Corollary. The three conics S_1, S_2, S have a common self conjugate triangle

EXERCISE VII. f.

1. If two conics S_1, S_2 are such that S_1 is its own reciprocal w r t S_2, prove that S_1, S_2 have double contact with each other

2. **If a rectangular hyperbola is reciprocated w.r t. a point O, prove that O lies on the director circle of the reciprocal conic.**
What special case arises if O lies on the rectangular hyperbola ?

3. **Reciprocate w.r.t. H the orthocentre H of a triangle inscribed in a rectangular hyperbola lies on the curve.**

4. What is the reciprocal of a system of conics passing through four fixed points w r t a vertex of the common self conjugate triangle ?

5. PQ is a variable chord of a rectangular hyperbola, subtending a right angle at a fixed point O, not on the curve , prove that PQ envelopes a parabola, having O as focus

6. A variable conic touches three fixed lines, and its director circle passes through a fixed point , prove that the conic touches another fixed line

7. Prove that the reciprocal of a parabola w.r.t. a parabola is a conic with one asymptote parallel to the axis of the second parabola.

8. If a fixed line meets a system of concentric circles, prove that the tangents at the points of intersection envelope a parabola.

9. Prove that the envelope of chords of an ellipse which subtend a right angle at the centre is a concentric circle.

10. (1) Prove that a conic reciprocated w.r.t. O becomes a rectangular hyperbola if, and only if, O lies on its director circle.

(2) Prove that any two conics can be reciprocated into rectangular hyperbolas.

(3) Prove that the director circles of a system of conics touching four straight lines are coaxal.

11. p is a variable tangent to a fixed conic; P is the centre of the circle which is the inverse of p w.r.t. a fixed point O and fixed radius of inversion k; prove that the locus of P is a conic.

12. Prove that the reciprocal of a hyperbola, eccentricity e, w.r.t. a parabola having a common focus and directrix, is an ellipse of eccentricity $\frac{1}{e}$ with the same focus and directrix.

13. Two conics S_1, S_2 have the same asymptotes OA, OB; the polar of a variable point on S_1 w.r.t. S_2 meets OA, OB at P, Q; prove that the triangle OPQ is of constant area.

14. A hyperbola and a parabola have a common focus and touch one another, and their common chord, length $2c$, passes through the focus; if $2l$ = the latus rectum of the parabola, prove that the eccentricity e of the hyperbola is given by $e^2 = 5 \pm 4\sqrt{\left(\frac{l}{c}\right)}$

15. With the focus of a hyperbola as centre, a circle is drawn touching the asymptotes; P is the pole w.r.t. the hyperbola of a tangent to the circle; if this tangent meets the directrix at Q, prove that PQ touches the circle.

16. The in-centre of a triangle self-conjugate to a hyperbola is at one of the foci; if e is the eccentricity and l the semi-latus rectum, prove that the in-radius $= \frac{l}{\sqrt{(e^2-2)}}$

17. A system of conics have a common focus and a corresponding directrix; prove that the normals to the conics at the extremities of the latera recta through the common focus touch a fixed parabola.

18. Find the condition that $xy = c^2$ may be its own reciprocal w.r.t. $x^2 + y^2 = a^2$.

19. A conic S is the polar reciprocal of itself w.r.t. another conic S'. Prove that the conics touch at two distinct points P, Q; that any chord of S through the pole of PQ is divided harmonically by S'; and that S' is the polar reciprocal of itself w.r.t. S.

CHAPTER VIII

HOMOGRAPHIC RANGES AND PENCILS

ALTHOUGH a knowledge of the fundamental property of the cross-ratio of a pencil of four concurrent lines dates back to Pappus, the general theory of ranges and pencils is essentially modern, and may be regarded as starting with Desargues and culminating in the comprehensive *Géometrie Supérieure* of Chasles, published in 1852, a treatise of remarkable originality. Chasles' discovery of the double points of cobasal homographic ranges led him to an ingenious method of solving a wide group of constructions, and his general theory of involution has afforded a new means of introducing imaginary elements into pure geometry. To weigh, however, the influence of Chasles on the progress of geometrical research, it is necessary to take into account his invaluable historical investigations of the work done in former centuries, contained in his *Aperçu Historique.*

For convenience of reference, we shall first enumerate certain standard theorems on cross-ratios. [Durell's *Modern Geometry*, pp. 65-75.]

(1) The cross-ratio of the range formed by four collinear points A, B, C, D is

$$\{ABCD\} = \frac{AB \cdot CD}{AD \cdot CB}.$$

The cross-ratio of the pencil formed by four concurrent lines a, b, c, d is

$$\{abcd\} = \frac{\sin \widehat{ab} \cdot \sin \widehat{cd}}{\sin \widehat{ad} \cdot \sin \widehat{cb}}.$$

(2) If the joins of corresponding points of two 4-point ranges are concurrent, the ranges are equi-cross.

If the meets of corresponding rays of two 4-ray pencils are collinear, the pencils are equi-cross.

(3) If two equi-cross 4-point ranges, on different bases, have a self-corresponding point, the joins of the other corresponding points are concurrent.

If two equi-cross 4-ray pencils, with different vertices, have a self-corresponding ray, the meets of the other corresponding rays are collinear.

(4) {ABCD} is unaltered in value, if, when any two letters are interchanged, the other two letters are also interchanged.

$\{abcd\}$ is unaltered in value, if, when any two letters are interchanged, the other two letters are also interchanged.

(5) If {ACBD} = {ADBC}, the range {AB ; CD} is harmonic.

If $\{acbd\} = \{adbc\}$, the pencil $\{ab\,;\ cd\}$ is harmonic.

(6) If A, B, C are three fixed collinear points, and if k is a constant, then there is *one and only one* point X such that {ABCX} $= k$.

If a, b, c are three fixed concurrent lines, and if k is a constant, then there is *one and only one* line x such that $\{abcx\} = k$.

Analytical Treatment.

(I) A, B, C are three fixed points on a base l; A′, B′, C′ are three fixed points on a base l'; O, O′ are fixed points, taken as origins, on l, l'. X, X′ are variable points on l, l' respectively such that {ABCX} = {A′B′C′X′}. It is required to find the relation connecting the positions of X, X′.

Let $\mathrm{OA} = a$, $\mathrm{OX} = x$; $\mathrm{O'A'} = a'$, $\mathrm{O'X'} = x'$; etc.

Now $\dfrac{\mathrm{AB}\,.\,\mathrm{CX}}{\mathrm{AX}\,.\,\mathrm{CB}} = \dfrac{\mathrm{A'B'}\,.\,\mathrm{C'X'}}{\mathrm{A'X'}\,.\,\mathrm{C'B'}}$, given; but $\mathrm{CX} = x - c$, etc.

$\therefore\ k\,.\,\dfrac{x-c}{x-a} = k'\,.\,\dfrac{x'-c'}{x'-a'}$, where k, k' are constants.

$$\therefore\ pxx' + qx + rx' + s = 0,$$

where p, q, r, s are constants; which is the required relation.

(II) O, O′ are fixed origins on the fixed lines l, l; X, X′ are variable points on l, l' subject to the condition

$$pxx' + qx + rx' + s = 0,$$

where $\mathrm{OX} = x$, $\mathrm{O'X'} = x'$; p, q, r, s being constants such that $ps \neq qr$. Then the range formed by any four positions of X is equi-cross with the range formed by the four corresponding positions of X′.

Let x_1, x_2, x_3, x_4 be the coordinates of any four points on l, and let x_1', x_2', x_3', x_4' be the corresponding points on l'.

Now $\{x_1x_2x_3x_4\} = \dfrac{(x_2-x_1)(x_4-x_3)}{(x_4-x_1)(x_2-x_3)}$; also $x_1 = -\dfrac{rx_1'+s}{px_1'+q}$, given.

$$\therefore\ x_2 - x_1 = \frac{rx_1'+s}{px_1'+q} - \frac{rx_2'+s}{px_2'+q} = \frac{(x_2'-x_1')(ps-qr)}{(px_1'+q)(px_2'+q)}.$$

Similarly, $\quad x_4 - x_3 = \dfrac{(x_4'-x_3')(ps-qr)}{(px_3'+q)(px_4'+q)}$, etc.

Therefore simplifying, since $ps - qr \neq 0$, we have

$$\frac{(x_2-x_1)(x_4-x_3)}{(x_4-x_1)(x_2-x_3)} = \frac{(x_2'-x_1')(x_4'-x_3')}{(x_4'-x_1')(x_2'-x_3')};$$

$$\therefore\ \{x_1x_2x_3x_4\} = \{x_1'x_2'x_3'x_4'\}.$$

Q.E.D.

Note. The condition $ps - qr \neq 0$ secures that the given relation $pxx' + qx + rx' + s = 0$ does not break up into linear factors.

But if $ps = qr$,

$$pxx' + qx + rx' + s \equiv \frac{1}{p}(p^2xx' + pqx + prx' + ps)$$

$$\equiv \frac{1}{p}(p^2xx' + pqx + prx' + qr) \equiv \frac{1}{p}(px+r)(px'+q).$$

And in this case, if $x = -\dfrac{r}{p}$, x' can have any value at all;

and if $x \neq -\dfrac{r}{p}$, x' must equal $-\dfrac{q}{p}$.

The relation between X, X′ is therefore no longer (1, 1).

Definition. If A, B, C, ... X, ...; A′, B′, C′, ... X′, ... are two ranges of points on the same or different bases, and if the cross-ratio of any four points of one range is equal to that of the four corresponding points of the other, the two ranges are said to be **homographic.**

We may state the results of (I) and (II) as follows:

The existence of a relation of the form $pxx' + qx + rx' + s = 0$, where p, q, r, s are constants, subject to $ps \neq qr$ is the necessary and sufficient condition that two ranges, generated from it, are homographic.

The fundamental characteristic of the homographic relation is the fact that it sets up a one-to-one correspondence. To any point of either range, there corresponds one and only one point of the other range.

Definition. If two homographic ranges have a common base, a point on the base which corresponds to itself for the two ranges is called a double point.

(III) Two homographic ranges on the same base have always two double points, which may be real, coincident or imaginary. Further, if there are more than two double points, then every point is a double point, and the ranges are identical.

Let the relation be $pxx' + qx + rx' + s = 0$.

For a double point, $x = x'$.

$$\therefore\ px^2 + qx + rx + s = 0 \quad \text{or} \quad px^2 + (q+r)x + s = 0.$$

This is a quadratic, and therefore has two roots, which may be real, coincident or imaginary.

Therefore there are two double points.

If, however, there are more than two double points, the quadratic is satisfied by more than two values of x, and so each coefficient must be zero.

$\therefore\ p = 0,\ q + r = 0,\ s = 0$, and the relation becomes

$$q(x - x') = 0,$$

in which case the ranges are identical. Q.E.D.

(IV) If one of the double points is at infinity, the ranges are **similar**, *i.e.* the line joining any two points is proportional to the line joining the corresponding points.

By hypothesis $x \to \infty$ satisfies $px^2 + (q+r)x + s = 0$.

Put $\xi = \dfrac{1}{x}$, then $\xi \to 0$ satisfies $p + (q+r)\xi + s\xi^2 = 0$;

$$\therefore\ p = 0.$$

$\therefore$ the relation reduces to $qx + rx' + s = 0$.

If y, y' ; z, z' are two pairs of corresponding points,

$$qy + ry' + s = 0 \quad \text{and} \quad qz + rz' + s = 0.$$

$\therefore$ subtracting, $\quad q(y - z) + r(y' - z') = 0$;

$$\therefore\ \frac{y - z}{y' - z'} = -\frac{r}{q} = \text{constant}. \qquad \text{Q.E.D.}$$

Corollary. If two ranges on the same base are similar, the point at infinity on the base is a double point.

Definitions.

(1) If two systems of concurrent lines are so related that the cross-ratio of any four rays of one system is equal to the cross-ratio of the corresponding rays of the other, then the two systems are said to form **homographic pencils.**

(2) If two homographic pencils have a common vertex, a line through the vertex which corresponds to itself for the two pencils is called a **double ray.**

Let $y=m_1x$, $y=m_2x$, $y=m_3x$ and $Y=m_1'X$, $Y=m_2'X$, $Y=m_3'X$ be two sets of three concurrent lines, referred to the same or different axes. To a variable line $y=mx$ of the first set corresponds the line $Y=m'X$ of the second set given by $\{m_1m_2m_3m\}=\{m_1'm_2'm_3'm'\}$. Then, as with ranges, we find that $pmm'+qm+rm'+s=0$, where p, q, r, s are constants. This sets up a (1, 1) correspondence between m, m'.

Conversely, by the same algebra as for ranges, we see that if m_1, m_1'; m_2, m_2'; m_3, m_3'; m_4, m_4' are any four pairs of solutions of the given relation

$$pmm'+qm+rm'+s=0, \quad ps \neq qr,$$

then $$\{m_1m_2m_3m_4\}=\{m_1'm_2'm_3'm_4'\}.$$

Hence, as before, we see that:

The existence of a relation of the form $pmm'+qm+rm'+s=0$, where p, q, r, s are constants subject to $ps \neq qr$, is the necessary and sufficient condition that two pencils, generated from it, are homographic.

(V) Two homographic pencils with the same vertex have always two double rays, which may be real, coincident or imaginary. Further, if there are more than two double rays, the two pencils are identical.

The proof is similar to that of (III).

(VI) An angle POP′ of constant magnitude rotates about a fixed vertex O; then the rays OP, OP′ generate homographic pencils, having as double rays the isotropic lines through O.

Let $y=mx$, $y=m'x$ be two positions of OP, OP′.

Then $\dfrac{m-m'}{1+mm'}=c$, where c is constant, given;

$$\therefore\ cmm'-m+m'+c=0.$$

$\therefore$ the rays generate homographic pencils.

To obtain the double rays, put $m=m'$;

$$\therefore\ cm^2+c=0 \text{ or } m^2=-1, \text{ since } c\neq 0.$$

$\therefore$ $m=\pm i$, and the double rays are $y=\pm ix$. Q.E.D.

Corollary. If the isotropic lines are the double rays of two homographic pencils with the same vertex, then corresponding rays are inclined at a constant angle.

It is merely necessary to reverse the order of the argument used to prove (VI).

EXERCISE VIII. a.

1. A, A′; B, B′; C, C′ are pairs of corresponding points on a line through O; OA $=1$, OB $=2$, OC $=3$; OA′ $=1$, OB′ $=3$, OC′ $=6$. Prove that the corresponding homographic relation is $xx'+9x-7x'-3=0$, and determine the double points.

2. Two homographic ranges are defined by $xx'+x-x'+1=0$; determine points in the second range corresponding to $x=-1, +1, 0, \infty$, and determine points in the first range corresponding to $x'=2, -2, 3, \infty$. Verify the cross-ratio property in each case.

3. P, P′ are a variable pair of corresponding points of the ranges defined by $pxx'+qx+rx'+s=0$, whose bases are l, l'. The point I on l corresponds to ∞ on l' and the point J′ on l' corresponds to ∞ on l; prove that PI . P′J′ is constant.

4. Two homographic ranges on the same base are defined by

$$xx'-x-4x'+6=0,$$

referred to the same origin; a is any point on the base at distance ξ from the origin; A′, A are the two points which correspond to a according as it is regarded as a point of the x-range or the x'-range; prove that $\text{A}'\text{A}=\dfrac{3(\xi-2)(\xi-3)}{(\xi-1)(\xi-4)}$. What can you say about the points for which $\xi=2$ and $\xi=3$? Also about the points for which $\xi=1$ and $\xi=4$?

5. Two homographic ranges whose bases are the x-axis and y-axis are defined by $xy - x - 4y + 6 = 0$. Find the points P, Q' corresponding to the origin for the two ranges. If I, J' correspond to the points at infinity on the two bases, prove that PQ' is parallel to IJ'.

6. Prove that a variable tangent to a parabola generates similar ranges on two fixed tangents. [With the fixed tangents as axes, the parabola may be written $\sqrt{ax} + \sqrt{by} = 1$.]

7. The relation defining two homographic pencils is

$$mm' + m - m' + 1 = 0;$$

find the four rays corresponding to $x = 0$, $y = 0$, $y = x$, $y = -x$, and prove that they form a harmonic pencil.

8. If the cross-ratio of the lines $y = mx$, $y = ix$, $y = m'x$, $y = -ix$ is k, prove that the angle between $y = mx$, $y = m'x$ is $\frac{1}{2i} \log k$ radians

9. If $u_1 \equiv a_1x + b_1y + c_1$, $u_2 \equiv a_2x + b_2y + c_2$, prove that the pair of lines $u_1 - \lambda k u_2 = 0$, $u_1 - \lambda u_2 = 0$, where λ varies, generate homographic pencils. What are the double rays?

10. What relations connect p, q, r, s if the isotropic lines are the double rays of the pencils defined by $pmm' + qm + rm' + s = 0$?

Geometrical Treatment. Corresponding to each perspective property of homographic ranges, there exists a dual theorem for homographic pencils. Since the proof can be effected by the ordinary verbal alterations (see p. 113), we shall enunciate the theorem and leave the reader to supply the proof.

Theorem 79. A, B, C are three fixed points on a base l; A′, B′, C′ are three other fixed points on a base l'. It is possible to construct in *one and only one way* pairs of points P, P′; Q, Q′; ... on l, l' such that the cross-ratio of any four of the points A, B, C, ... P, Q, ... is equal to that of the corresponding points of the range A′, B′, C′, ... P′, Q′, ...

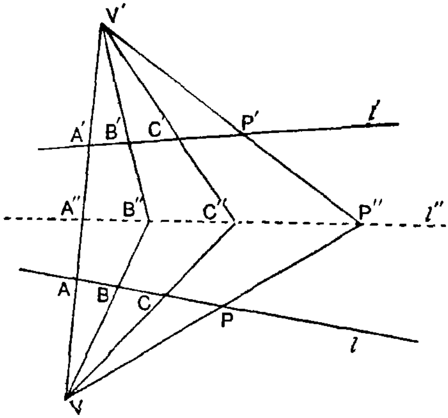

FIG. 70

Take any two points V, V′ on AA′; B″, C″ are the meets of VB, V′B′; VC, V′C′; and l'' is the join of B″C″.

Take any point P on l: let VP meet l'' at P″, and let V′P″ meet l' at P′. Similarly, if Q, R, S, ... are any points on l, construct the corresponding points Q″, R″, S″, ... on l'' and Q′, R′, S′, ... on l'.

Then $\{PQRS\} = V\{PQRS\} = \{P''Q''R''S''\} = V'\{P''Q''R''S''\}$
$= \{P'Q'R'S'\}$.

The construction is therefore always possible.

But, by (6) on p. 130, for any given point P there exists only one point P′ such that $\{A'B'C'P'\} = \{ABCP\}$. Therefore the construction is unique. Q.E.D.

Theorem 80. a, b, c are three fixed lines through a vertex L; a', b', c' are three other fixed lines through a vertex L'. It is possible to construct in *one and only one way* pairs of lines p, p'; q, q'; ... through L, L' such that the cross-ratio of any four of the lines $a, b, c, \dots p, q, \dots$ is equal to that of the corresponding lines of the pencil $a', b', c', \dots p', q', \dots$.

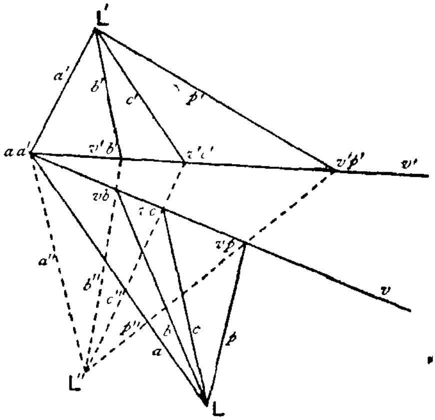

FIG 71.

This is the dual of Theorem 79; the figure has been lettered so that it corresponds in every respect to that of Theorem 79.

Theorem 81. (1) If two ranges $\{A, B, \dots P, \dots\}$, $\{A', B', \dots P', \dots\}$, are each homographic to a third range $\{A_1, B_1, \dots P_1, \dots\}$, then they are homographic to each other.

(2) If two pencils $\{a, b, \dots p, \dots\}$, $\{a', b', \dots p', \dots\}$ are each homographic to a third pencil $\{a_1, b_1, \dots p_1, \dots\}$, then they are homographic to each other.

Since $\{PQRS\} = \{P_1Q_1R_1S_1\}$ and $\{P'Q'R'S'\} = \{P_1Q_1R_1S_1\}$,

$$\therefore \{PQRS\} = \{P'Q'R'S'\}.$$

$\therefore$ the ranges $\{A, \dots P, \dots\}$, $\{A', \dots P', \dots\}$ are homographic.

A similar proof applies to the dual theorem. Q.E.D.

Theorem 82. Homographic ranges reciprocate into homographic pencils; and conversely.

This follows at once from the fact that the cross-ratio of four collinear points is equal to that of their polars.

Note. Theorems 79, 80 remain true if the bases l, l' or the vertices L, L′ coincide. Thus, if l, l' coincide, it is only necessary to take an auxiliary base l_1 (preferably through A) and construct on it a range $A_1, B_1, C_1, \dots P_1, \dots$ homographic to A, B, C, … P, … , and then construct on l' the range A′, B′, C′, … P′, … homographic to $A_1, B_1, C_1, \dots P_1, \dots$.

If L, L′ coincide, a dual method may be used, taking an auxiliary vertex L_1 (preferably on a).

We may now re-state Theorems 79, 80 as follows:

Theorem 79. (*Alternative Form.*) Homographic ranges on the same or different bases exist, and are determined uniquely when three given points on one base correspond respectively to three given points on the other base.

Theorem 80. (*Alternative Form.*) Homographic pencils with the same or different vertices exist, and are determined uniquely when three given lines through one vertex correspond respectively to three given lines through the other vertex.

Definitions.

(1) Two ranges are said to be in **perspective** if the joins of corresponding points are concurrent.

(2) Two pencils are said to be in **perspective** if the meets of corresponding rays are collinear.

(3) Two ranges (A), (A′) are said to be **projective** if one or more ranges (A_1), (A_2), (A_3), .. can be found such that each of the ranges (A), (A_1), (A_2), (A_3), … (A′) is in perspective with the range following it in the sequence.

(4) Two pencils (a), (a') are said to be **projective** if one or more pencils (a_1), (a_2), (a_3), … can be found such that each of the pencils (a), (a_1), (a_2), (a_3), … (a') is in perspective with the pencil following it in the sequence.

Theorem 83. (1) Projective ranges are homographic, and conversely, homographic ranges are projective.

(2) Projective pencils are homographic, and conversely, homographic pencils are projective.

These results follow at once from Theorems 79, 80.

Theorem 84. (1) If two homographic ranges on different bases have one self-corresponding point, they are in perspective.

(2) If two homographic pencils with different vertices have one self-corresponding ray, they are in perspective.

These results follow at once from (3) on p. 130.

Theorem 85. If two homographic ranges $\{A, B, \ldots P, Q, \ldots\}$, $\{A', B', \ldots P', Q', \ldots\}$ have different bases l, l', then the meet of PQ', $P'Q$ lies on a fixed straight line.

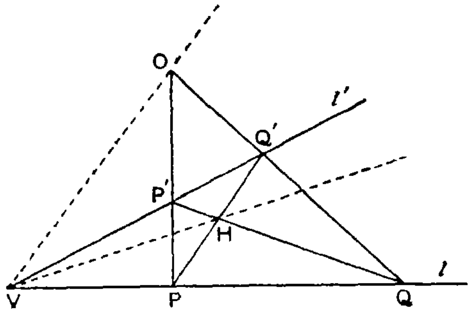

FIG 72

Case I. Suppose the ranges are in perspective, so that AA', BB', ... PP', ... meet at a point O.

Let V be the meet of l, l' and H the meet of PQ', $P'Q$.

By the harmonic property of the quadrilateral, $V\{PP'; OH\}$ is harmonic; but VP, VO, VP' are fixed lines.

$\therefore$ VH is a fixed line. Q.E.D.

Case II. If the ranges are not in perspective, the meet of l, l' is not a self-corresponding point in the two ranges. Denote the meet of l, l' by C or D′, according as it is regarded as belonging to (A) or (A′); C′, D denote the points corresponding to C, D′.

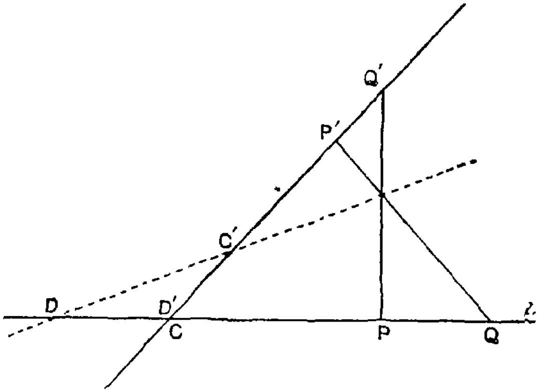

FIG. 73.

Now, by hypothesis, P{P′Q′C′D′} = P′{PQCD}.

These have a self corresponding ray PP′; therefore the meets of PQ′, P′Q; PC′, P′C; PD′, P′D are collinear. But the meet of PC′, P′C is C′, and the meet of PD′, P′D is D.

∴ the meet of PQ′, P′Q lies on the fixed line C′D. Q.E.D.

Theorem 86. If two homographic pencils $\{a, b, .. p, q, ..\}$, $\{a', b', ... p', q', ..\}$ have different vertices L, L′, then the join of pq', $p'q$ passes through a fixed point.

This is the dual of Theorem 85; the proof of Case I presents no difficulty In order to assist the reader in the proof of Case II we append the figure, lettered in dual fashion.

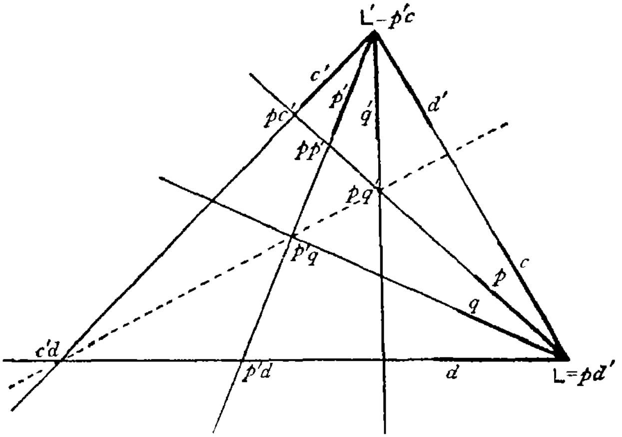

FIG. 74.

Definitions.

(1) If {A, P, Q, }, {A′, . P′, Q′ } are two homographic ranges on different bases, the straight line which is the locus of the meet of PQ′, P′Q is called the **cross-axis** of the two ranges.

(2) If $\{a, .p, q, \}$, $\{a', p', q', \}$ are two homographic pencils with different vertices, the fixed point which lies on the join of pq', $p'q$ is called the **cross-centre** of the two ranges.

These names are due to Dr Filon the alternative terms employed are "*homographic axis*" and "*homographic centre.*"

EXERCISE VIII. b.

1. A, B are fixed points; P is a variable point on a fixed line; AP, BP meet another fixed line in P_1, P_2; prove that P_1, P_2 generate homographic ranges, and find two positions in which P_1, P_2 coincide.

2. O{A, B, ... P, ...}, O′{A, B′, ... P′, ...} are two homographic pencils; any two fixed lines AX, AX′ cut OP, O′P′ at P, P′; prove that PP′ passes through a fixed point.

3. The sides QR, RP, PQ of a variable triangle pass through fixed points A, B, C; P, Q move on fixed straight lines; prove that AR, BR generate homographic pencils

4. A, B are two fixed points; a variable line through A meets two fixed lines at P_1, P_2; prove that BP_1, BP_2 generate homographic pencils, and determine the two positions in which BP_1, BP_2 coincide.

5. A, B, H, K are four fixed points; X is a variable point on HK; AX, BX cut a fixed line at A′, B′; HA′, KB′ meet at Y; prove that the locus of Y is a straight line concurrent with AB, A′B′.

6. PQR is a variable triangle of given shape inscribed in a fixed triangle; prove that its vertices generate homographic ranges on the sides of the fixed triangle

7. A variable conic touches the sides AB, AC of a given triangle ABC and cuts BC at two fixed points; prove that its points of contact with AB, AC generate homographic ranges. [Use projection.]

8. A variable conic touches four fixed lines; prove that its points of contact generate homographic ranges. [Project the dual property.]

Notation. If {A, B, ... P, ... }, {A′, B′, ... P′, ... } are two homographic ranges on the same or different bases l, l', we shall denote the points at infinity on l, l' by J, I′ and the points corresponding to them on l', l by J′, I.

If l, l' coincide, we shall denote the double points by E, F; similarly, we shall denote the double rays of two homographic pencils with the same vertex by e, f.

Theorem 87. *(1) If P, P′ are a variable pair of corresponding points of two homographic ranges, then* PI . P′J′ *is constant.*

(2) Conversely, if I, J′ *are fixed points on the fixed lines l, l', and if* P, P′ *are a variable pair of points on l, l' such that* PI . P′J′ *is constant, then* P, P′ *generate homographic ranges.*

(1) By hypothesis, $\{PIQJ\} = \{P'I'Q'J'\}$;

$$\therefore \frac{PI \cdot QJ}{PJ \cdot QI} = \frac{P'I' \cdot Q'J'}{P'J' \cdot Q'I'}, \quad \text{but } \frac{QJ}{PJ} = 1 = \frac{P'I'}{Q'I'};$$

$$\therefore \frac{PI}{QI} = \frac{Q'J'}{P'J'} \quad \text{or} \quad PI \cdot P'J' = QI \cdot Q'J';$$

$\therefore$ PI . P′J′ is constant. Q.E.D.

(2) This follows at once by reversing the order of argument in (1).

Corollary. *If l, l' coincide, and if* E *is a point on l such that* EI . EJ′ = AI . A′J′, *then* E *is a double point of the two ranges.*

Construction for I, J′. Given three pairs A, A′; B, B′; C, C′ of corresponding points on a common base l, to construct I, J′.

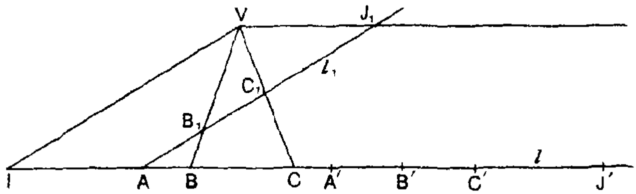

FIG. 75.

Draw through A any line l_1, and cut off AB_1, AC_1 equal to A′B′, A′C′; let BB_1, CC_1 meet at V.

Draw VJ_1, VI parallel to l, l_1 respectively to meet l_1, l at J_1, I. Cut off A′J′ on l equal to AJ_1; then I and J′ are the required points.

We have $\{ABC\infty\} = V\{ABC\infty\} = V\{AB_1C_1J_1\} = \{AB_1C_1J_1\}$
$= \{A'B'C'J'\}$.

Also $\{ABCI\} = V\{ABCI\} = V\{AB_1C_1\infty\} = \{AB_1C_1\infty\}$
$= \{A'B'C'\infty\}$. Q.E.D.

Theorem 88. If the lines IA, J′A′ are rotated in *opposite* directions about I, J′ through a right angle into the positions Ia, J′a', then the

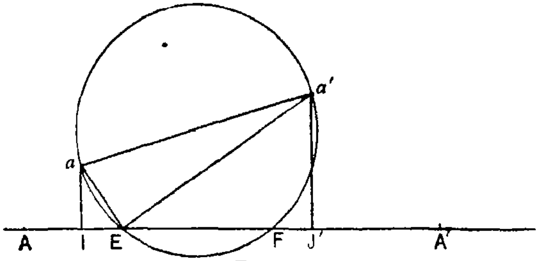

FIG. 76.

circle on aa' as diameter cuts the common base at two points E, F (real, coincident or imaginary), which are the double points for the two ranges.

Join Ea, Ea'. Since aa' is a diameter, $\angle a\text{E}a' = 90°$.

∴ the triangles IaE, J′Ea' are similar;

$$\therefore \frac{\text{EI}}{\text{I}a} = \frac{a'\text{J}'}{\text{EJ}'};$$

$$\therefore \text{EI} \,.\, \text{EJ}' = \text{I}a \,.\, a'\text{J}' = \text{AI} \,.\, \text{A}'\text{J}', \text{ given.}$$

∴ by Theorem 87 Corollary, E is a double point; similarly, F is also a double point. Q.E.D.

Note. The reader should also draw a figure in which the order of points on l is A, I, A′, J′; in this case he will observe that a, a' fall on opposite sides of l, so that the double points are necessarily real.

This construction is due to Professor A. Lodge.

If A, A′ and I and J′ are given, the figure of Theorem 88 gives a simple method for constructing any corresponding pair of points P, P′ of the ranges.

Take any point π on the circle; then $a\pi$, $a'\pi$ meet the base l at a corresponding pair of points P, P′; for the triangles aIP, P′J′a' are similar;

$$\therefore \frac{\text{PI}}{a\text{I}} = \frac{\text{J}'a'}{\text{P}'\text{J}'} \quad \text{or} \quad \text{PI} \,.\, \text{P}'\text{J}' = a\text{I} \,.\, \text{J}'a' = \text{AI} \,.\, \text{A}'\text{J}',$$

so that P, P′ correspond.

Theorem 89. (1) If E, F are the double points of the cobasal homographic ranges {A, . . P, . . }, {A′, . . P′, . . }, then {PEP′F} is constant.

(2) Conversely, if E, F are two fixed points, and if P, P′ are a variable pair of points on EF such that {PEP′F} is constant, then P, P′ generate homographic ranges with E, F as double points.

(1) By hypothesis, {PEQF} = {P′EQ′F}.

$$\therefore \frac{\mathrm{PE}}{\mathrm{PF}} \cdot \frac{\mathrm{QF}}{\mathrm{QE}} = \frac{\mathrm{P'E}}{\mathrm{P'F}} \cdot \frac{\mathrm{Q'F}}{\mathrm{Q'E}} \quad \text{or} \quad \frac{\mathrm{PE}}{\mathrm{PF}} \cdot \frac{\mathrm{P'F}}{\mathrm{P'E}} = \frac{\mathrm{QE}}{\mathrm{QF}} \cdot \frac{\mathrm{Q'F}}{\mathrm{Q'E}};$$

$$\therefore \{\mathrm{PEP'F}\} = \{\mathrm{QEQ'F}\};$$

$$\therefore \{\mathrm{PEP'F}\} \text{ is constant.} \qquad \text{Q.E.D.}$$

(2) To prove this, it is merely necessary to reverse the order of the argument in (1).

Theorem 90. (1) If e, f are the double rays of the homographic pencils $\{a, \ldots p, \ldots\}$, $\{a', \ldots p' \ldots\}$, which have a common vertex, then $\{pep'f\}$ is constant.

(2) Conversely, if e, f are two fixed lines and p, p' a variable pair of lines through ef, such that $\{pep'f\}$ is constant, then p, p' generate homographic pencils having e, f as double rays.

This is the dual of Theorem 89.

Theorem 91. If the points at infinity on the bases l, l' of two homographic ranges correspond, then the ranges are similar.

By hypothesis, $\{\mathrm{PQR}\infty\} = \{\mathrm{P'Q'R'}\infty\}$,

$$\therefore \frac{\mathrm{PQ}}{\mathrm{RQ}} = \frac{\mathrm{P'Q'}}{\mathrm{R'Q'}} \quad \text{or} \quad \frac{\mathrm{PQ}}{\mathrm{P'Q'}} = \frac{\mathrm{QR}}{\mathrm{Q'R'}}. \qquad \text{Q.E.D.}$$

Corollary. If two ranges are similar, the points at infinity on their bases correspond to each other.

Theorem 92. If {A, B, ... }, {A′, B′, .. } are two homographic cobasal ranges, there exist two positions of a point L such that the pencils L{A, B, .. }, L{A′, B′, ... } can be superposed, *i.e.* corresponding rays are inclined to each other at a constant angle: and these two positions of L are real if, and only if, the double points of the ranges are imaginary.

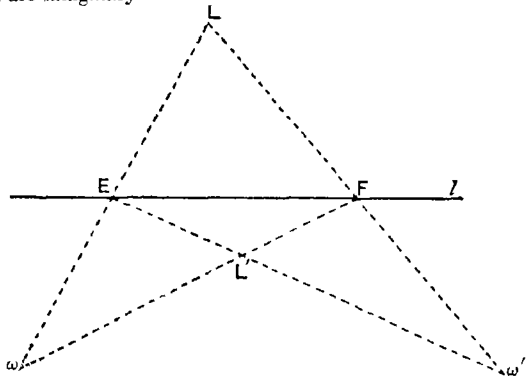

FIG. 77.

Let E, F be the double points, and let ω, ω' be the circular points at infinity. Let Eω, Fω' meet at L and Eω', Fω meet at L′; then the isotropic lines are the double rays of the homographic pencils L{A, B, . .}, L{A′, B′, ...}.

$\therefore$ L{PωP′ω'} is constant; $\therefore$ $\angle$PLP′ is constant.

$\therefore$ L is one of the required points; similarly, L′ is the other.

Further, if E, F are imaginary points, they are conjugate imaginaries, and therefore the lines Eω, Fω' are conjugate imaginary lines, so that their meet L is real; and similarly, L′ is real.

But if E, F are real, there cannot be any other real point on the lines Eω, Fω'; therefore L is imaginary: similarly with L′; and it is clear that in this case L, L′ are conjugate imaginaries. Q.E.D.

EXERCISE VIII. c.

1. Two homographic cobasal ranges are defined by the pairs $x=0$, $x'=\frac{3}{4}$; $x=6$, $x'=0$; $x=-2$, $x'=\frac{4}{5}$; referred to the same origin. Find the coordinates of I, J′, E, F.

2. IJ′ is a diameter of a circle, centre O; a variable tangent to the circle cuts the tangents at I and J′ to the circle in P, P′, prove that (i) the triangles OIP, P′J′O are similar, (ii) IP J′P′ is constant, (iii) P, P′ trace out homographic ranges

3. Prove that $\mathrm{EI}=\mathrm{J'F}$.

4. If O′ is the point of the (A′) range which corresponds to the mid-point O of IJ′ regarded as belonging to the (A) range, prove that

$$\mathrm{OE}^2=\mathrm{OF}^2=\mathrm{IO} \quad \mathrm{OO'}=\mathrm{OJ'}\,.\,\mathrm{OO'}$$

Deduce that the double points are real if O does not lie between J′ and O′

5. Two homographic ranges on the x axis are determined by

$$xx'-2ax+a^2+b^2=0\ ,$$

find the coordinates of the two positions of L of Theorem 92.

6. Repeat No 5 for the relation $xx'-2ax+a^2-b^2=0$.

7. Prove that two homographic pencils with a common vertex can be projected into pencils in which corresponding rays are inclined at a constant angle.

8. A, B are two fixed points, P, P′ are a pair of variable points on AB such that $\frac{\mathrm{AP}}{\mathrm{PB}} \quad \frac{\mathrm{AP}}{\mathrm{P'B}}$ is constant Prove that P, P′ generate homographic ranges Determine the double points.

9. What is the homographic relation if the double points coincide at the origin ?

10. Prove that if A, A′ are fixed, AP J′P′ is proportional to A′P′

11. Prove that $\frac{\mathrm{AI}}{\mathrm{AP}}+\frac{\mathrm{A'J'}}{\mathrm{A'P'}}=1$.

12. Prove that $\frac{\mathrm{AP}}{\mathrm{A'P'}} \quad \mathrm{BC}+\frac{\mathrm{BP}}{\mathrm{B'P'}}\,.\,\mathrm{CA}+\frac{\mathrm{CP}}{\mathrm{C'P'}} \quad \mathrm{AB}=0$.

13. If the ranges {A, B, }, {A′, B′, } are connected by the relation $xx'+qx+rx'+s=0$, referred to A, B′ as origins respectively, prove that $q=-\mathrm{B'J'}$, $r=-\mathrm{AI}$, $s=\mathrm{AI}\,.\,\mathrm{B'A'}$.

14. A, B are two fixed points on a circle Σ, P is a variable point on Σ: prove that AP, BP generate homographic pencils Determine the ray in the pencil, vertex A, corresponding to the ray BA in the pencil vertex B.

15. A variable tangent to a fixed circle, centre O, cuts two fixed tangents to the circle at P_1, P_2; prove that (i) $\angle P_1OP_2$ is constant, (ii) P_1, P_2 generate homographic ranges What points in these ranges correspond to the meet of the two fixed tangents ?

16. Prove that $EI.EP' + EP.EJ' = EP.EP'$

Constructions by Trial and Error. (Second Degree.)

Example. [*Poncelet's Problem.*] To describe a quadrilateral such that its four sides pass through given points and its four corners lie on given lines.

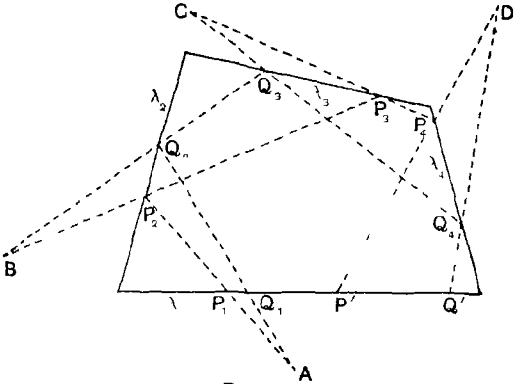

FIG 78.

Let A, B, C, D be the four fixed points and λ_1, λ_2, λ_3, λ_4 the four fixed lines

Through A draw any line cutting λ_1, λ_2 at P_1, P_2, join BP_2 and produce it to cut λ_3 at P_3, join CP_3 and produce it to cut λ_4 at P_4 join DP_4 and produce it to cut λ_1 at P′

Similarly, take any number of lines through A and construct the sets of points Q_1, Q_2, Q_3, Q_4, Q′, R_1, R_2, R_3, R_4, R′;

Then $\{P_1Q_1, \quad\} = A\{P_1, Q_1, \quad\} = \{P_2, Q_2, \quad\} = B\{P_2, Q_2 \quad\}$
$= \{P_3, Q_3, \quad\} = C\{P_3, Q_3, \quad\} = \{P_4, Q_4, \quad\}$
$= D\{P_4, Q_4, \quad\} = \{P', Q', \ .\}$,

$\therefore$ $\{P_1, Q_1, \ldots\}$, $\{P', Q, \quad\}$ are homographic ranges on λ_1.

Let E, F be the double points of the two ranges. Then either AE or AF may be taken as a side of the required quadrilateral, the remaining sides are then at once determined. Q.E.D.

The method is general, and applies with equal ease to any n-sided polygon whose sides pass through fixed points and whose n corners lie on a fixed line. The practical application of this method will provide the reader with an exercise in drawing of by no means a simple character. We give below the *complete* construction for

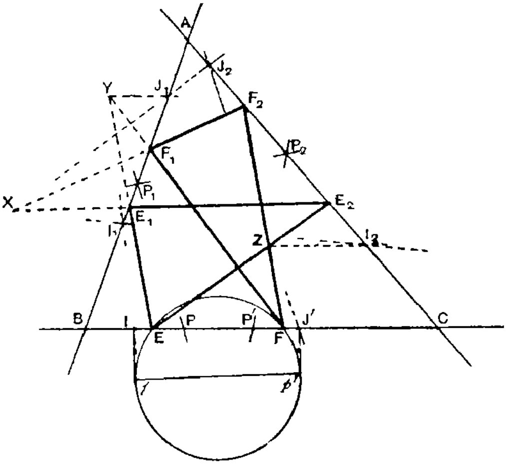

FIG 79

Poncelet's Problem in the case of a triangle, the fixed points being X, Y, Z, the reader should examine its details. The work is shortened by constructing I and J′ direct, as shown in Fig. 79.

EXERCISE VIII. d.

1. {A, B, }, {A′, B′, } are given homographic ranges on different bases l, l', through a given point O, construct a line to cut l, l' at corresponding points Is there more than one solution ?

2. Through a given point, draw a line to cut two given lines at points subtending a given angle at a given point

3. Find two points P, Q on two fixed lines such that PQ subtends angles of given size at each of two fixed points

4. Through a given point draw two lines to cut off on two given lines segments of given lengths

5. Describe a triangle PQR so that its sides pass through given points, and so that P, Q lie on fixed lines and $\angle$ PRQ is of given magnitude.

6. A, B are fixed points; l, m are fixed lines; find a point P on l such that PA, PB cut off from m a segment of given length.

7. Inscribe in a given triangle a rectangle of given area.

8. Show how to determine a pair of parallel straight lines which pass through fixed points and cut two fixed lines in points collinear with a given point.

9. Describe a quadrilateral so that its four corners lie on fixed lines, two of its sides pass through fixed points and the other two are in a given direction.

10. A ray of light starts from a given source and is reflected successively at n fixed lines; if its final path makes a given angle with its initial path, construct its initial path.

CHAPTER IX

HOMOGRAPHIC PROPERTIES OF THE CONIC

IF V_1, V_2, A, B, C, D, E, are a system of points on a conic, the pencils $V_1\{A, B, \}$, $V_2\{A, B, . \}$ are homographic. It is therefore unnecessary to specify the particular position of the point V on the conic, when dealing with cross-ratio properties of the pencil

$$V\{A, B, \}$$

In the same way, if v_1, v_2, $a, b, c, d, e,$ are a system of tangents to a conic, the ranges $v_1\{a, b, \}$, $v_2\{a, b, \}$ are homographic, and so it is unnecessary to specify the particular position of the tangent v to the conic.

Definitions.

(1) A system of points A, B, on a conic is called a **range of points on the conic** or a **range of the second order.**

(2) A system of tangents $a, b,$ to a conic is called a **pencil of tangents to the conic** or a **pencil of the second order.**

(3) If two ranges of points on a conic A, B, . . , A′, B′, are such that the pencils $V\{A, B, \}$, $V\{A', B', . \}$ are homographic, where V is any other point on the conic, the ranges are said to be **homographic.**

(4) If two pencils of tangents to a conic $a, b,$; $a', b',$ are such that the ranges $v\{a, b, \}$, $v\{a', b', \}$ are homographic, where v is any other tangent to the conic, the pencils are said to be **homographic.**

Theorem 93. (1) Two homographic ranges of points on a conic exist and are determined uniquely, when three pairs of corresponding points are given

(2) Two homographic pencils of tangents to a conic exist and are determined uniquely, when three pairs of corresponding tangents are given

(1) Let A, A' B, B , C, C' be the given pairs of points Take any other point V on the conic Then, by Theorem 80, two homographic pencils exist and are determined uniquely by the pairs of rays VA, VA VB, VB , VC, VC Let any other pair of corresponding rays VP, VP of these pencils cut the conic at P P , then P, P are a pair of corresponding points of the homographic ranges of points on the conic Q E D

(2) This is the dual of (1) and is proved in the same way

Theorem 94 (1) If $\{A, B \quad P, Q, \quad\}$, $\{A , B , \quad P , Q , \quad\}$ are two homographic ranges of points on a conic, there exist two points E F (real coincident or conjugate imaginaries) on the conic which are self corresponding for the two ranges

(2) The meet of PQ', $P Q$ lies on the fixed line EF

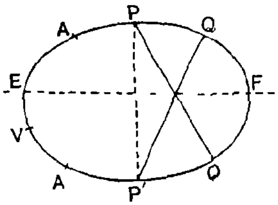

FIG 80

(1) Take any point V on the conic, the pencils $V\{A \; B, \quad\}$, $V\{A \; B', \quad\}$ have two double rays, let the other points of intersection of these double rays with the conic be E, F Then E, F are self corresponding points for the two ranges of points on the conic

(2) Since E, F are self corresponding points,

$$P\{P'EFQ\} = P\{PEFQ\}$$

But these pencils have a self corresponding ray PP', therefore the meets of PE $P E$ PF $P F$, PQ', $P Q$ are collinear

the meet of PQ , $P Q$ lies on the fixed line EF

This line is necessarily real, because it contains any number of real points, such as the meet of AB and $A B$ Therefore its points of intersection E, F with the conic are either real, coincident or conjugate imaginaries Q E.D.

Theorem 95. (1) If $\{a, b, \quad p, q, \quad\}$, $\{a', b', \quad p', q', \quad\}$ are two homographic pencils of tangents to a conic, there exist two tangents e, f (real, coincident or conjugate imaginaries) to the conic, which are self corresponding for the two pencils

(2) The join of pq, $p'q$ passes through the fixed point ef

This is the dual of Theorem 94, the reader should draw his own figure and go through the proof

Definitions.

(1) With the notation of Theorem 94, E, F are called the **double points** of the two homographic ranges of points on the conic, and the line EF is called the **cross-axis** of the two ranges

(2) With the notation of Theorem 95, e, f are called the **double lines** of the two homographic pencils of tangents to the conic, and the point ef is called the **cross-centre** of the two pencils

It should be noted that Pascal's theorem is a special case of Theorem 94

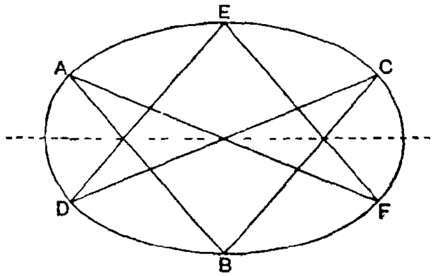

FIG 81

Let ABCDEF be a hexagon inscribed in a conic and consider the homographic ranges of points on the conic determined by {A, C, E} and {D, F, B}. Then the meets of AB, DE, BC, EF, CD, FA lie on a straight line, the cross axis

The reader is advised to deduce in the same way Brianchon's theorem from Theorem 95.

Theorem 96. (1) Homographic ranges of points on a conic project into homographic ranges of points on the projected conic, double points into double points and the cross-axis into the cross axis.

(2) Homographic pencils of tangents to a conic project into homographic pencils of tangents to the projected conic, double lines into double lines and the cross-centre into the cross centre.

These results follow from the fact that cross ratios are unaltered by projection.

Theorem 97. Homographic ranges of points on a conic reciprocate into homographic pencils of tangents to the reciprocal conic, double points into double lines, and the cross axis into the cross centre; and conversely.

These results follow from the fact that the cross ratio of four collinear points is equal to that of their polars.

Theorem 98. (1) If {A, B, C, ...} is a range of points on a conic, and if $\{a, b, c, ...\}$ is the pencil of the tangents to the conic at these points, the range {A, B, C, ...} and the pencil $\{a, b, c, ...\}$ are homographic, i.e. any section of V{A, B, C, ...} is homographic to the range $v\{a, b, c, ...\}$.

(2) If {A, B, C, ...} is a range of collinear points, and if $\{a, b, c, ...\}$ is the pencil of concurrent lines formed by their polars w.r.t. any conic, the range A, {B, C, ...} and the pencil $\{a, b, c, ...\}$ are homographic, i.e. any section of the pencil is homographic to the range.

These results follow at once from Theorems 30, 28

Double-Point Construction. The property of the cross-axis in Theorem 94 affords another method of determining the double points of two cobasal homographic ranges.

Let {A, B, ... }, {A′, B′, ... } be the two ranges. Draw any circle or conic, and take any point V on it: let VA, VA′, VB, VB′, ... cut

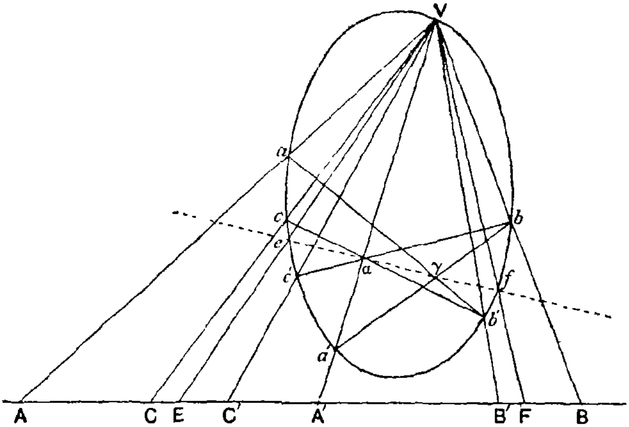

FIG. 82.

the conic again at a, a', b, b', ... ; let ab', $a'b$ meet at γ ; bc', $b'c$ meet at α ; let $\alpha\gamma$ cut the conic at e, f. Join Ve, Vf, and produce them to meet the base of the given ranges at E, F.

Then E, F are the required double points. Q.E.D.

EXERCISE IX. a.

1. The base of a variable triangle inscribed in a given conic is fixed, prove that the sides generate homographic pencils

2. One vertex of a triangle, self conjugate w r t a given conic, is fixed, prove that the other vertices generate homographic ranges

3. AP, AP are a variable pair of conjugate lines through a fixed point A w r t a given conic, prove that they generate homographic pencils, and determine the double rays

4 APQ is a triangle inscribed in a given conic A is fixed and ∠PAQ is of constant size prove that the tangents at P, Q generate homographic pencils of tangents to the conic

5 Two conics S_1, S_2 have double contact at B C, A is the pole of BC, tangents are drawn from a variable point P on AB to S_1, S_2 to cut AC at P_1, P_2, prove that P_1, P_2 generate homographic ranges, and determine the double points

6. The sides QR, RP, PQ of a triangle pass through fixed points A, B, C, P lies on a fixed conic through B, C, Q lies on a fixed conic through C, A, prove that BR, AR generate homographic pencils

7 AB is a fixed chord of a circle, PQ is a variable chord of constant length, prove that AP, BQ generate homographic pencils

8. A variable line passes through a fixed point, prove that its poles w r.t. two given conics generate homographic ranges

9. Two conics cut at B, C a variable line through B cuts the conics at P, Q, prove that CP, CQ generate homographic pencils

10. A, B are fixed points on a hyperbola, P is a variable point on the curve, PA, PB meet an asymptote at P_1, P_2, prove that P_1, P_2 generate homographic ranges Where are the double points? Deduce that P_1P_2 is of constant length

11. A, B are fixed points on a parabola, P is a variable point on the curve, parallels to PA, PB are drawn through a fixed point O to cut a fixed diameter in P_1 P_2, prove that P_1, P_2 generate homographic ranges and that P_1P_2 is of constant length

12 Given five points A, B, C, D, E, show how to determine the points in which the conic through A, B, C, D, E cuts a given line l [Consider the range formed on l by A{C, D, E}, B{C, D, E}]

13. HK is a fixed diameter of a given conic, a variable tangent meets the tangents at H, K in P, P′, prove that HP KP is constant [Use Theorem 87]

14 ABCD is a fixed parallelogram, circumscribing a given conic, a variable tangent cuts AB, AD at P, Q, prove that BP DQ is constant

15 A variable pair of conjugate diameters of a given conic meet the tangent at a fixed point P in Q, Q′; prove that Q, Q′ generate homographic ranges, and deduce that PQ PQ′ is constant

Theorem 99. (1) If {A, B, … }, {A′, B′, … } are two homographic ranges of points on a conic, the joins of AA′, BB′, … envelope a conic having double contact with the given conic at the double points E, F of the two ranges.

(2) Conversely, if two conics have double contact at E, F, a variable tangent to one cuts the other at a pair of points P, P′, which generate homographic ranges of points on the conic, with E, F as double points.

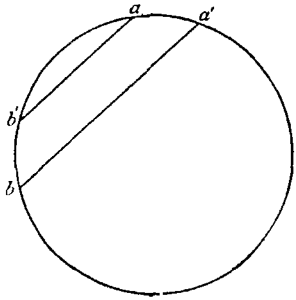

FIG 83

(1) Project E, F into the circular points at infinity Then the conic becomes a circle, and the cross axis of the projected ranges $\{a, b, \ldots\}$, $\{a', b', \ldots\}$ is the line at infinity

∵ ab', $a'b$ are parallel, and so chord aa' = chord bb'.

∴ $aa' = bb' = cc' = \ldots$, ∴ the lines aa', bb', cc', … touch a concentric circle

∴ in the original figure AA′, BB′, … touch a conic having double contact with the given conic at E, F. Q E D.

(2) This is proved by reversing the order of argument in (1).

Corollary. If a variable chord PP′ of a conic passes through a fixed point O, then P, P′ generate homographic ranges of points on the conic, having as double points the points of contact of the tangent from O to the conic

To prove this, project the conic into a circle, having the projection of O as centre.

Theorem 100. If $\{a, b, \ldots\}$, $\{a', b', \quad\}$ are two homographic pencils of tangents to a conic, the meets aa', bb', of corresponding tangents lie on a conic having double contact with the given conic at the points of contact with the conic of the double lines e, f of the two pencils · and *conversely*

Corollary. From a variable point P on a fixed line, tangents p_1, p_2 are drawn to a given conic, then p_1, p_2 generate homographic pencils of tangents to the conic.

This is simply the dual of Theorem 99

EXERCISE* IX. b.

1. H is a point of intersection of two given conics S_1, S_2, a variable line through H cuts S_1, S_2 at P_1, P_2, prove that P_1 and P_2 generate homographic ranges of points on S_1 and S_2

What is the dual theorem ?

2. B, C are a fixed pair of conjugate points w r t a given conic S, P is a variable point on S, BP, CP meet S again at P_1, P_2, prove that P_1, P_2 generate homographic ranges of points on S Determine the double points

3. A variable conic passes through four fixed points, prove that the tangents at these points generate homographic pencils

4. A, B are two fixed points, PAQ is a variable chord of a given conic, BP, BQ meet the conic again at P, Q′, prove that P′, Q′ generate homographic ranges on the conic

5. A system of conics pass through four fixed points A, B are any two other points prove that the polars of A w r t the system of conics form a pencil homographic to the polars of B w r t the system

6. AB, AC are a fixed pair of conjugate lines w r t a conic S, a variable tangent to S cuts AB AC at P, P′, prove that the other tangents from P, P′ to S generate homographic pencils of tangents to S

7. PQR is a variable triangle, inscribed in a given conic, PQ, PR pass through fixed points, find the envelope of QR.

8. PQRS is a quadrilateral circumscribing a conic, P, Q, R lie on fixed lines, find the locus of S.

9. A variable triangle is inscribed in a given conic, two of the sides are fixed in direction, find the envelope of the third side

10. A variable triangle circumscribes a conic; two of its vertices lie on fixed lines; prove that the points of contact of its sides generate three homographic ranges of points on the conic.

11. Two conics S_1, S_2 have double contact with each other; the tangents from a variable point on S_1 to S_2 meet S_1 again at P, Q; prove that P, Q generate homographic ranges on S_1.

12. ABC is a fixed triangle inscribed in a given conic; PQ is a variable chord such that A{BPCQ} is constant; find the envelope of PQ.

13. A, B, C, D and p, q, r, s are the common points and common tangents of two conics; prove that the range of points A, B, C, D on one conic is equicross with the range of tangents p, q, r, s to the other.

14. Two conics S_1, S_2 have double contact at A, B; a variable chord PQ of S_1 touches S_2; find the locus of the meet of AP, BQ.

Theorem 43 and Theorem 44 are most useful when stated in terms of homographic ranges and pencils, as follows:

(i) If two pencils V{A, B, ... P, ...}, W{A, B, ... P, ...} are homographic but not in perspective, the meets A, B, ... P, ... of corresponding rays lie on a conic through the vertices V, W of the pencils.

(ii) If two ranges {A, B, ... P, ...}, {A′, B′ ... P′, ...} are homographic but not in perspective, the joins AA′, BB′, ... PP′, ... of corresponding points envelope a conic, touching the bases α, α' of the ranges.

Theorem 101. [Apollonius' Theorem] The feet of the four normals to a conic, centre C, from any point O lie on a rectangular hyperbola which passes through C, O and has its asymptotes parallel to the axes of the conic.

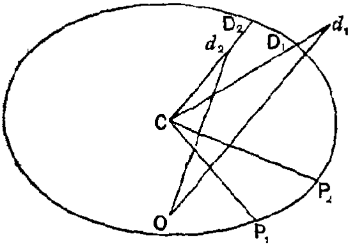

FIG 84

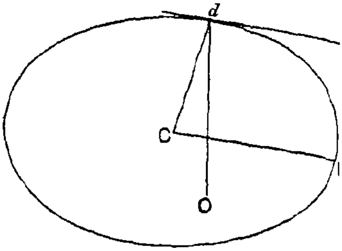

FIG 85

Let CP_1, CD_1, CP_2, CD_2, be pairs of conjugate diameters of the conic Let perpendiculars from O to CP_1, CP_2, meet CD_1, CD_2, at d_1 d_2,

Then $O\{d_1, d_2, \} = C\{P_1\ P_2, \}$, equiangular pencils,
$= C\{D_1, D_2, \}$, since the diameters are conjugate,
$= C\{d_1, d_2\ \}$

the points d_1, d_2 lie on a conic σ through O, C

If d is a point of intersection of σ with the given conic, since Cd is conjugate to CP, the tangent at d is parallel to CP, and is therefore perpendicular to Od Therefore d is the foot of one of the normals from O to the given conic

σ meets the given conic at the feet of the normals from O

Let CA, CB be the axes of the given conic , then the perpendicular from O to CA meets the conjugate diameter CB at infinity

Therefore σ passes through the point at infinity on CB and similarly through the point at infinity on CA, and so σ is a rectangular hyperbola passing through O, C and having its asymptotes parallel to CA, CB

Q E D

This rectangular hyperbola is called the **hyperbola of Apollonius**, the method of proof is due to Chasles

EXERCISE IX. c.

1. P is a variable point on a given conic S, Q is a point such tha PQ subtends angles of given sizes at two fixed points on S, find th locus of Q.

2. P is a variable point on a fixed diameter of a conic, N is the foo of the perpendicular from P to its polar, prove that the locus of N is a rectangular hyperbola

3 {A, B, P, }, {A′, B′, P′, } are similar ranges on different bases, prove that PP′ envelopes a parabola

4 ABC is a triangle, A is fixed, B moves on a fixed line and ∠ABC is of constant size, find the envelope of BC

5. H K are the poles of the chords PQ, RS of a conic, prove that H, K, P, Q, R, S lie on a conic

6. A triangle circumscribes a given conic, two of its sides are fixed in position, find the locus of its circumcentre

7. HK is a fixed chord of a parabola, a variable line PQ perpendicular to the axis cuts HK at P, the polar of P cuts PQ at Q, find the locus of Q

8 A variable chord PQ of a fixed conic passes through a fixed point O, prove that the polar of O w r t the circle on PQ as diameter envelopes a parabola

9. If a quadrangle is inscribed in a conic prove that the tangents at its vertices and one pair of opposite sides touch a conic

10 AB is a given chord of a fixed conic, the tangent at a variable point P on the conic meets the tangent at A in Q, find the envelope of the line through Q, parallel to BP

11 P is a variable point on a fixed line, PQR is a line fixed in direction, cutting two fixed lines at Q, R, X is the harmonic conjugate of P w r t Q, R, prove that the locus of X is a hyperbola, having one asymptote parallel to the fixed direction

12 A variable line l meets two fixed lines in points which are conjugate w r t a given conic, find the envelope of l

13. A is the pole of a fixed chord BC of a given conic, two variable parallel tangents to the conic cut AB, AC at P, Q respectively, find the envelope of PQ

14. D is a variable point on the base BC of a given triangle ABC, a parallel through D to AB cuts AC at E, a parallel through E to BC cuts AD at M, prove that the locus of M is a parabola through C, touching AB at A with its axis parallel to BC

15. A straight line passes through a fixed point, prove that the join of its poles w r t two given conics envelopes a conic, inscribed in the common self conjugate triangle of the given conics

16. PQR is a variable triangle inscribed in a fixed circle; PQ is of constant length and PR passes through a fixed point. Find the envelope of QR.

17. 123456 is a hexagon inscribed in a conic; prove that the points 1; 2; (13, 24); (23, 41); (15, 26); (25, 16) lie on a conic.

18. A tangent at a variable point P on a parabola meets a fixed tangent at Q; find the locus of a point dividing PQ in a constant ratio.

19. P, P′ are a pair of corresponding points of two homographic ranges on different bases; find the locus of the mid-point of PP′.

20. P, Q, R, S are the feet of the four normals from a point to a conic; prove that the tangents at P, Q, R, S touch a parabola which touches the axes of the conic. [Use reciprocation.]

21. With a fixed point O as centre, circles are described to cut a given conic in 4 points. Prove that the diagonal points of this 4-point quadrangle lie on the Apollonian hyperbola of O.

22. If the polar of O w.r.t. a conic Σ meets the Apollonian hyperbola of O w.r.t. Σ at Q, R, prove that

(i) $\angle$ QOR = 90°; (ii) $\triangle$ QOR is self-conjugate w.r.t. Σ.

23. If, in No. 20, O is the point from which the normals are drawn, and if C is the centre of the conic, prove that CO is the directrix of the parabola.

CHAPTER X

INVOLUTION RANGES AND PENCILS

Analytical Treatment.

If two homographic ranges are situated on the same base, to every point ξ of that base there correspond in general two distinct points, according as ξ is regarded as belonging to the first or second range. Under certain conditions, however, these two points coincide for *all* positions of ξ.

(I) If $pxx' + qx + rx' + s = 0$, $ps \neq qr$, is the homographic relation between two ranges on the same base, referred to the same origin, then $q = r$ is the **necessary and sufficient** condition that to any point α on the base there corresponds the same point β, whichever range α belongs to.

(i) Let $x = \alpha$, $x' = \beta$ and $x' = \alpha$, $x = \beta$ for all values of α.

$\therefore$ $p\alpha\beta + q\alpha + r\beta + s = 0$ and $p\beta\alpha + q\beta + r\alpha + s = 0$.

$\therefore$ subtracting, $(\alpha - \beta)(q - r) = 0$.

But $\alpha \neq \beta$, for in this case $p\alpha^2 + (q + r)\alpha + s = 0$, which cannot be true for all values of α.

$$\therefore\ q - r = 0 \quad \text{or} \quad q = r.$$

The condition is therefore necessary.

(ii) If $q = r$, the relation becomes $pxx' + qx + qx' + s = 0$.

Hence, if $x = \alpha$, $x' = -\dfrac{q\alpha + s}{p\alpha + s}$, and if $x' = \alpha$, $x = -\dfrac{q\alpha + s}{p\alpha + s}$.

Therefore the two points corresponding to the point α on the base coincide. Therefore the condition is sufficient.

Definition. If two cobasal ranges are determined by the relation $pxx' + q(x + x') + s = 0$, $ps \neq q^2$, referred to the same origin, they are said to be **in involution.**

Suppose that {A, B, C, . }, {A′, B′, C′, ... } are two ranges in involution, since the relation is homographic, the cross-ratio of any four points of one range is equal to that of the corresponding four points of the other range

But further, if A is now regarded as belonging to the second range, the point corresponding to it in the first range, by **(I)**, is A′, and similarly for B, C, ...

It therefore follows that the ranges {A, A′, B, B′, C, C′, }, {A′, A, B , B, C′, C, } are homographic **and this is the characteristic feature of two homographic ranges** {A}, {A′} **in involution** Thus, for example, {AA′BC} = {A′AB C } or {ABB′C } = {A′B BC}

(II) A range of points in involution exists and is determined uniquely by two pairs of points on the base

This follows from the fact that the involution relation contains two *independent* constants $\frac{q}{p}, \frac{s}{p}$

Assuming that $p \neq 0$, the involution relation may be written

$$xx' + \frac{q}{p}(x + x') + \frac{s}{p} = 0 \quad \text{or} \quad \left(x + \frac{q}{p}\right)\left(x' + \frac{q}{p}\right) - \frac{q^2 - ps}{p^2}$$

Now change the origin to the point $x = -\frac{q}{p}$, the relation becomes

$$XX' = \frac{q^2 - ps}{p^2}$$

If $X \to 0$, $X' \to \infty$, so that the new origin corresponds to the point at infinity on the base

Definition. The point on the base which corresponds to the point at infinity on the base, in an involution range is called the **centre of the involution**, and will always be denoted by O

The involution relation shows that if P, P′ are a pair of corresponding points, then OP OP′ is constant

If $X = X' = \xi$, say, we see that $\xi^2 = \frac{q^2 - ps}{p^2}$ or $\xi = \pm \frac{1}{p}\sqrt{(q^2 - ps)}$

Consequently two ranges in involution have two **double points**, real or imaginary, which are equidistant from the centre of the involution.

The two double points will always be denoted by E, F

Since $OE^2 = OF^2 = \frac{q^2 - ps}{p^2} = OP \cdot OP'$, it follows that every pair of corresponding points are harmonically conjugate to the double points.

The Case where p = 0. In this case the involution-relation reduces to $q(x + x') + s = 0$ or $\frac{x + x'}{2} = -\frac{s}{2q}$.

Hence, if P, P′ are any pair of corresponding points, the mid point of PP′ is fixed and its coordinate is $-\frac{s}{2q}$. The involution therefore consists of pairs of points which are equidistant from a fixed point E.

It therefore follows that E is one double point, the point at infinity is the other double point, and the centre of the involution is also the point at infinity on the base.

EXERCISE X. a.

1. If the involution relation is $xx' - 3x - 3x' + 8 = 0$, find the points corresponding to $x = 0, 1, 2, 3, 4, \infty$.

By a change of origin, reduce the relation to the form $XX' = c^2$

2. Determine the involution defined by the point pairs (1, 2); (5, 7) Find the double points, and reduce the relation to its simplest form.

3. Repeat No. 2 for the point pairs (1, 10); (4, 7).

4. Find a if the point pairs (2, 5), (1, 8); (3, a) are in involution

5. Determine the involution relation if the double points are given by $\xi^2 + 2a\xi + b = 0$

6. A, A′; B, B′, C, C′ are point-pairs in involution, α, β, γ are the mid points of AA′, BB′, CC′; prove that $\frac{AB \cdot AB'}{AC \cdot AC'} = \frac{\alpha\beta}{\alpha\gamma}$. [Take A as origin]

7. With the notation of No. 6, prove that

$$\frac{AB \cdot AB'}{A'B \cdot A'B'} = \frac{AC \cdot AC'}{A'C \cdot A'C'}.$$

8. Prove that the necessary and sufficient condition that the three pairs of points given by

$$a_1\xi^2 + 2b_1\xi + c_1 = 0, \quad a_2\xi^2 + 2b_2\xi + c_2 = 0, \quad a_3\xi^2 + 2b_3\xi + c_3 = 0$$

are in involution, is

$$a_1(b_2c_3 - b_3c_2) + b_1(c_2a_3 - c_3a_2) + c_1(a_2b_3 - a_3b_2) = 0.$$

Write this as a determinant.

9. Prove that the double points of the involution defined by the point-pairs $\xi^2+2b_1\xi+c_1=0$, $\xi^2+2b_2\xi+c_2=0$, are given by

$$x^2(b_1-b_2)+x(c_1-c_2)=b_1c_2-b_2c_1.$$

10. An involution is defined by the point pairs $a_1\xi^2+2b_1\xi+c_1=0$, $a_2\xi^2+2b_2\xi+c_2=0$, prove that any other point-pair of the involution is represented by $a_1\xi^2+2b_1\xi+c_1+\lambda(a_2\xi^2+2b_2\xi+c_2)=0$, for a suitable value of λ

Pencils in Involution. The analogy between homographic ranges and pencils subsists naturally between involution ranges and pencils.

(III) If $pmm'+qm+rm'+s=0$, $ps\neq qr$, is the homographic relation between two pencils with a common vertex, then $q=r$ is the necessary and sufficient condition that to any line α through the vertex corresponds the same line β, whichever pencil α belongs to

Definition. If two pencils with a common vertex are connected by the relation $pmm'+q(m+m')+s=0$, $ps\neq q^2$, referred to the same axes, they are said to be **in involution.**

If $\{a, b, c, \quad\}$, $\{a', b', c', \quad\}$ are two pencils in involution, it follows from the definition and result (III) that the pencils

$$\{a, a', b, b', c, c', \quad\}, \quad \{a', a, b', b, c', c, \quad\}$$

are homographic **and this is the characteristic feature of two pencils** $\{a\}$, $\{a'\}$ **in involution.** Thus, for example, $\{aa'bc\}=\{a'ab'c'\}$ and $\{a'bb'c\}=\{abb'c'\}$

(IV) A pencil of lines in involution exists and is determined uniquely if any two pairs of corresponding lines through the vertex are given.

This is due to the fact that the involution relation contains two *independent* constants.

If we put $m=m'=\mu$, say, we see that

$$p\mu^2+2q\mu+s=0 \quad \text{or} \quad \mu=\frac{1}{p}\{-q\pm\sqrt{q^2-ps}\}.$$

Hence there are two **double lines,** real or imaginary. If $y=\mu x$ is a double ray, we have

$$p\cdot\frac{y^2}{x^2}+2q\cdot\frac{y}{x}+s=0 \quad \text{or} \quad sx^2+2qxy+py^2=0.$$

It is easy to see that any line-pair of an involution pencil harmonically conjugate to the double lines

Let $(y-mx)(y-m'x)=0$ or $mm'x^2-xy(m+m')+y^2=0$ be line pair, so that $pmm'+q(m+m')+s=0$; the double lines a $sx^2+2qxy+py^2=0$, and the condition for a harmonic pencil $pmm'+s=-q(m+m')$, which is satisfied.

EXERCISE X. b.

1. If the involution relation is $mm'-4m-4m'+15=0$, find the lin corresponding to $y=x$, $y=2x$, $y=3x$, $y=0$, $x=0$ Determine th double lines

2. Determine the involution defined by the line pairs $y=x$, $y=3x$ $y=4x$, $y-7x$ Find the double lines

3. Prove that $ax^2+2hxy+by^2=0$ is a line pair of the involutio $pmm'+q(m+m')+s=0$ if $ap+bs=2hq$

4 Determine the involution if the double lines are

$$\text{(i) } x^2-8xy+13y^2=0, \quad \text{(ii) } x^2+y^2=0$$

5. Prove that the line pairs

$$x^2+2h_1xy+b_1y^2=0, \quad x^2+2h_2xy+b_2y^2=0, \quad x^2+2h_3xy+b_3y^2=0$$

are in involution if $b_1(h_2-h_3)+b_2(h_3-h_1)+b_3(h_1-h_2)=0$

6 Prove that the double lines of the involution determined b $x^2+2h_1xy+b_1y^2=0$, $x^2+2h_2xy+b_2y^2=0$ are

$$(h_1-h_2)x^2+(b_1-b_2)xy+(b_1h_2-b_2h_1)y^2=0$$

7 Prove that, in every involution, there is one line pair at righ angles, and that if there is more than one such line pair then ever line pair is at right angles, and the double lines are the isotropic lines

8. Prove that any line pair of the involution defined by

$$a_1x^2+2h_1xy+b_1y^2=0, \quad a_2x^2+2h_2xy+b_2y^2=0$$

can be represented by

$$(a_1x^2+2h_1xy+b_1y^2)+\lambda(a_2x^2+2h_2xy+b_2y^2)=0,$$

for a suitable value of λ

9. Prove that the line pairs $s_1=0$, $s_2=0$, $s_1-\lambda s_2=0$, $s_1-\lambda' s_2=0$ $s_1-\mu s_2=0$, $s_1-\mu' s_2=0$ form an involution if $\lambda\lambda'=\mu\mu'$, where

$$s_1=a_1x+b_1y+c_1 \quad s_2=a_2x+b_2y+c_2$$

10. Prove that the double lines of the involution determined by

$$\phi_1\equiv a_1x^2+2h_1xy+b_1y^2=0, \quad \phi_2\equiv a_2x^2+2h_2xy+b_2y^2=0$$

are given by $\dfrac{\partial\phi_1}{\partial x}\dfrac{\partial\phi_2}{\partial y}=\dfrac{\partial\phi_2}{\partial x}\dfrac{\partial\phi_1}{\partial y}$.

Geometrical Treatment.

The analytical treatment was developed from the relation

$$pxx' + q(x + x) + r = 0,$$

which was shown to be reducible to $XX' = \lambda$. we shall now take the latter as our starting point

Definition. O is a fixed point on a fixed line A, A′, B, B′, C, C′, .. are point pairs on the line such that

$$OA \cdot OA' = \lambda = OB \cdot OB = OC \quad OC = \quad .$$

Then the system of point pairs A, A′, B, B, C, C, is said to **form an involution**, and O is called the **centre** of the involution.

From this definition, it follows that O corresponds to the point at infinity on the base

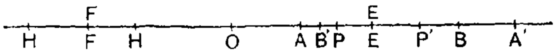

FIG 86.

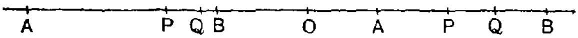

FIG 87

If λ is positive (Fig 86) O is external to the portion of the base joining each point pair, and the portion of the base joining each point pair lies wholly inside or wholly outside or wholly includes the portion of the base joining any other point pair, *i e* they do not overlap Whereas if λ is negative (Fig. 87), the portion of the line joining any point pair contains O and overlaps the portion of the base joining any other point pair

If λ is positive, the involution is called **non-overlapping** or **hyperbolic**, if λ is negative, the involution is called **overlapping** or **elliptic.**

This definition of involution directs attention to the fundamental characteristic that *the unit of an involution is a point pair, whereas the unit of homographic ranges is a point.*

Theorem 102. Given two point-pairs A, A′; B, B′ on a base l construct a point O on l such that OA . OA′ = OB . OB′.

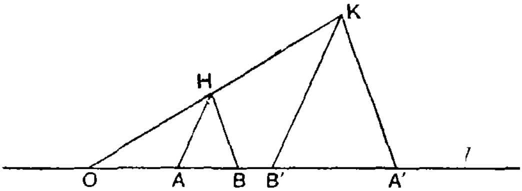

FIG. 88.

Through A, B′ draw any two parallel lines to cut any other two parallel lines through B, A′ respectively at H, K, produce HK to meet l at O.

By parallels, $$\frac{OA}{OB'} = \frac{OH}{OK} = \frac{OB}{OA'};$$

$$\therefore\ OA \cdot OA' = OB \cdot OB'. \qquad \text{Q.E.F.}$$

Theorem 103. If A, A′, B, B′, C, C′ are three point-pairs in involution, then {AA′BC} = {A′AB′C′}.

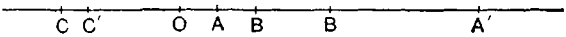

FIG 89

Let O be the centre of the involution so that

$$OA \cdot OA' = OB \cdot OB' = OC \cdot OC' = k;$$

let α be the point at infinity in the direction perpendicular to the base OA. Then αA′, αA, αB′, αC′ are the polars of A, A′, B, C w.r.t. the circle, centre O, radius $\sqrt{k}$

$$\therefore\ \{AA'BC\} = \alpha\{A'AB'C'\} = \{A'AB'C'\}. \qquad \text{Q.E.D.}$$

Theorem 104. Any two point pairs A, A′, B, B′ on a base l determine uniquely an involution

By Theorem 102, it is possible to find a point O on l such that OA . OA′ = OB . OB . If P is any point on l, and if P′ is another point on l such that OP . OP = OA . OA′, then P, P′ form a point pair in involution with A, A′, B, B′

And further, the position of P′ corresponding to P is unique, since, by Theorem 103, {A′ABP} = {AA′B′P }

Therefore there exists one and only one involution having A, A′, B, B′ as point pairs. Q E D

Corollary. Given t o point pairs A, A′, B, B on a base l, there exists one and only o‹ “point O on l such that OA OA = OB OB′

For O corresponds to the point at infinity on l in the unique involution defined by A, A′, B, B′

Theorem 105. If A, A , B, B , C C are three pairs of points on a base l such that {AA′BC} = {A′AB C }, then these three pairs form an involution

If possible, let C be the point corresponding to C in the involution defined by A, A′, B, B′

By Theorem 103, {AA′BC} = {A′AB′C″}

But {AA′BC} = {A′AB C }, given, ∵ {A′AB′C″} = {A′AB′C′},

∴ C coincides with C′ Q E D.

Theorem 106. The necessary and sufficient condition that the point-pairs A, A′; B, B′, C, C′, D, D′, form an involution is that the ranges

{A, A′, B, B , C, C′, D, D }, {A′, A, B′, B, C′, C, D′, D, }

should be homographic.

(i) The method of proof of Theorem 103 shows that the condition is necessary.

(ii) Theorem 105 shows that the condition is sufficient, for, if {AA′BP} = {A′AB′P }, P, P′ is a point-pair of the unique involution defined by A, A′, B, B′. Q E D

Definition. If a point on the base of an involution corresponds to itself, it is called a **double point** of the involution.

A double point of an involution is therefore a point-pair, the two elements of which coincide.

Theorem 107. (1) In any involution there exist two double points, equidistant from the centre of the involution, which are real or imaginary, according as the involution is hyperbolic or elliptic.

(2) Any point-pair of the involution is harmonically conjugate to the double points.

(3) An involution exists and is determined uniquely when the double points are given.

If P, P′ is any point-pair and O the centre of the involution

$$OP \cdot OP' = k.$$

Therefore there are two double points E, F given by $OE^2 = OF^2 = k$ or $OE = +\sqrt{k}$, $OF = -\sqrt{k}$.

E, F are real if k is positive, in which case the involution is non-overlapping or hyperbolic.

Further, since $OE^2 = OF^2 = k = OP \cdot OP'$, $\{PP' ;\ EF\}$ is harmonic.

Since O is the mid-point of EF and $k = OE^2$, the involution exists and is unique when E, F are given.

Corollary 1. If three pairs of points are each harmonically conjugate to two fixed points E, F, then they form an involution with E, F as double points.

Corollary 2. If three pairs of points are in involution, there exist two points w.r.t. which each pair is harmonically conjugate.

Theorem 108. If three point-pairs A, A′ ; B, B′ ; C, C′ are in involution, then $AB' \cdot BC' \cdot CA' = -A'B \cdot B'C \cdot C'A$; and conversely.

We have

$$\{AB'BC'\} = \{A'BB'C\} \quad \text{or} \quad \frac{AB' \cdot BC'}{AC' \cdot BB'} = \frac{A'B \cdot B'C}{A'C \cdot B'B};$$

$$\therefore\ AB' \cdot BC' \cdot CA' = -A'B \cdot B'C \cdot C'A.$$

The converse is proved by reversing the order of this argument.

Q.E.D.

Theorem 109. (1) Any transversal is cut by a system of coaxal circles in point-pairs in involution.

(2) P, P′ is a variable point-pair of a given involution; H is any fixed point outside the base; then the circle HPP′ belongs to a fixed coaxal system.

(1) Let the transversal cut the radical axis at O and any one circle at P, P′; then OP . OP′ is constant.

$\therefore$ P, P′ generate an involution.

(2) Let O be the centre of the involution so that $OP \,.\, OP' = k$; join OH, and let it cut the circle again at L; then

$$OH \,.\, OL = OP \,.\, OP' = k.$$

Therefore L is a fixed point. Therefore the circle passes through two fixed points, and so belongs to a fixed coaxal system. Q.E.D.

Theorem 110. A, B, C, . . are any system of points on a base l; A′, B′, C′ . . are the conjugate points on l, w.r.t. a given conic Σ, of A, B, C, ..., then the point-pairs A, A′; B, B′; C, C′; . . form an involution on l, having as double points the meets of l with Σ.

Let l meet Σ at α, β; then $\{AA'; \alpha\beta\}$ is harmonic, since A, A′ are conjugate points.

$\therefore$ by Theorem 107, Corollary 1, A, A′; B, B′, . . form an involution with α, β as double points. Q.E.D.

Theorem 111. If E, F are the double points of the involution defined by the point-pairs A, A′; B, B′; then the point pairs A, B; A′, B′; E, F form an involution.

By hypothesis, $\{AB'EF\} = \{A'BEF\} = \{BA'FE\}$

$\therefore$ A, B; A′ B′; E, F form an involution. Q.E.D.

Note. The same argument shows that A, B′; A′, B; E, F also form an involution.

Construction. Given two point-pairs A, A′ ; B, B′ ; construc the point C′ corresponding to a given point C so that A, A′ , B, B′ C, C′ form an involution

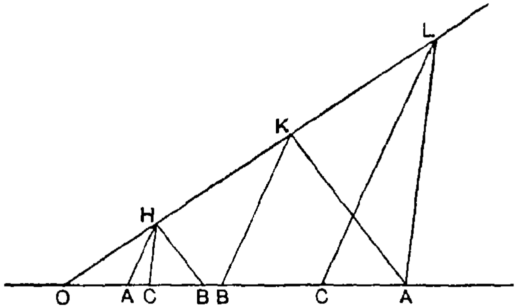

FIG 90.

First Method. Take any point H outside the base , draw throu B′, A′ lines parallel to AH, BH to meet at K , draw A′L parallel CH to meet HK at L ; draw LC′ parallel to AH to meet the base at (Then C′ is the required point.

Suppose HK meets the base at O, it is easy to prove by parall that OA OA —OB OB —OC OC .

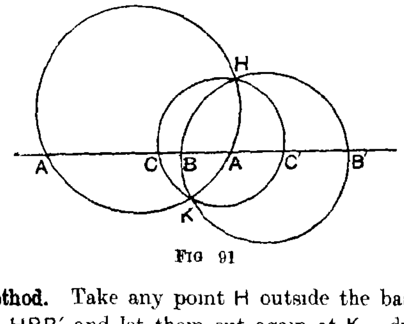

FIG 91

Second Method. Take any point H outside the base , draw circles HAA′, HBB′ and let them cut again at K , draw the ci CHK to cut the base at C′

Then, by Theorem 109 (2), C′ is the required point.

EXERCISE X. c.

1. If O is the centre of the involution A, A′; B, B′; prove that $\frac{OA}{OB}=\frac{AB'}{BA'}$, and that $\frac{AB\cdot AB'}{A'B\cdot A'B'}=\frac{AO}{A'O}$.

2. If A, A′; B, B′; C, C′, are point-pairs of an overlapping involution, prove that there exists a real point P at which AA′, BB′, CC′, subtend a right angle.

3. If AA′, BB′, CC′, . form an involution with E, F as double points, prove that

(i) $AB=\frac{AC}{A'C'}\cdot BC'+\frac{CB}{C'B'}\cdot C'A$, (ii) $\frac{BC\cdot BC'}{BA\cdot BA'}+\frac{AC\cdot AC'}{AB\cdot AB'}=1$,

(iii) $\frac{AP\cdot AP'}{A'P\cdot A'P'}$ is constant, (iv) $\frac{EP\cdot FP'}{PP'}$ is constant,

(v) $\frac{AB\cdot AB'}{AE^2}=\frac{A'B\cdot A'B'}{A'E^2}$ (vi) $\frac{AB}{BE}+\frac{AB'}{B'E}=\frac{AA'}{AE}$.

4. If $\{XY;PP'\}=\{XY,QQ'\}=\{XY,RR'\}=-1$, prove that
$$\{PP'QR\}=\{PP'R'Q'\}$$

5. Prove that two straight lines divided homographically can be placed, one on the other, so as to form an involution
[With the usual notation, place I on J′.]

6. If a variable line cuts three fixed circles in involution, prove that it passes through a fixed point, unless the circles are coaxal

7. Prove that any line through the cross-centre of two homographic pencils is cut by the pencils in involution.

8. E, F are the double points of the involution A, A′, B, B′, if BB′ is contained wholly in AA′, prove that AB, A′B′ subtend equal angles at any point on the circle on EF as diameter

9. ABC, A′B′C′ are two triangles such that BC, B′C′, CA, C′A′, AB, A′B′ intersect at three collinear points P, Q, R, AA′, BB′, CC′ meet the line PQR at P′, Q′, R′, prove that P, P′, Q, Q′, R, R′ form an involution

10. Prove that the double points of the involution made on any straight line by a coaxal system are concyclic with the limiting points

11. A system of conics have ABC as a common self-conjugate triangle; prove that any line through A is cut in involution by the conics

12. P, P′ is a point pair of an involution of which E, F are the double points, prove that the circle on PP′ as diameter is orthogonal to any circle through E, F

Involution Pencils. If a, a'; b, b'; $c, c', \ldots$ are a system of line-pairs, drawn through a point V, such that one (and therefore every) transversal is cut by them in an involution range, then the system is said to form an **involution pencil** or to be **in involution**; and V is called the **vertex** of the involution.

The involution pencil is called **elliptic** or **hyperbolic and overlapping** or **non-overlapping,** according as one (and therefore every) transversal is cut in an overlapping or non-overlapping involution.

If a line through the vertex of an involution pencil corresponds to itself, it is called a **double line** of the involution.

It is obvious that the double lines cut any transversal of the pencil at the double points of the involution formed on that transversal by the pencil.

If the line pairs of an involution are at right angles, the involution is said to be **orthogonal.**

The results given in Theorem 112 follow at once from corresponding properties of involution ranges.

Theorem 112. (1) An involution pencil exists and is determined uniquely by two pairs of lines a, a', b, b' which concur at a point V.

(2) The necessary and sufficient condition that the line-pairs a, a'; b, b'; c, c', ... form an involution is that the pencils

$$\{a, a', b, b', c, c', \ldots\}, \quad \{a', a, b', b, c', c, \ldots\}$$

should be homographic.

(3) Every involution pencil has two double lines, real or imaginary; each line pair of the pencil is harmonically conjugate to the double lines, and any pair of lines harmonically conjugate to the double lines form a line pair of the pencil.

(4) If three pairs of lines are each harmonically conjugate to two given lines, they form an involution having these two given lines as double lines.

(5) $a, b, c, \ldots$ are any system of lines through a vertex V; $a', b', c', \ldots$ are the conjugate lines through V w.r.t. a given conic Σ of $a, b, c, \ldots$; then the line pairs a, a', b, b'; c, c'; ... form an involution having as double lines the tangents from V to Σ.

Theorem 113. (1) Every involution pencil has one line-pair at right angles.

(2) If more than one line-pair is at right angles, then every line-pair is at right angles, and the double lines of the pencil are the isotropic lines.

(1) The internal and external bisectors of the angle between the double lines are at right angles, they are also harmonically conjugate to the double lines, and therefore form a line-pair of the pencil

(2) Let VA, VA′, VB, VB′ be two perpendicular line pairs These determine an involution pencil uniquely, but each line pair is harmonically conjugate to the isotropic lines $V\omega$, $V\omega'$, since

$$\angle AVA' = 90^\circ = \angle BVB'.$$

$V\omega$, $V\omega'$ are the double lines of the pencil defined by VA, VA′, VB VB′ If then VP, VP′ is any other line pair of the pencil, $V\{PP', \omega\omega'\}$ is harmonic, and therefore $\angle PVP' = 90^\circ$. Q E D

Theorem 114. If the isotropic lines through the vertex are a line pair of the pencil, then the double lines are at right angles, and conversely.

This follows at once from Theorem 112 (3)

Theorem 115. If the angles formed by each of two line pairs of an involution have the same bisectors, then these bisectors are the double lines of the involution, and the angle formed by any other line pair has the same bisectors

This follows at once from Theorem 112 (1) and (3)

Theorem 116. If A, A′, B, B′, C, C′, form an involution, there exist two positions of a point V such that the involution pencil V{A, A′, B, B′, C, C′, } is orthogonal, and these positions are real if the involution is elliptic and are conjugate imaginaries if the involution is hyperbolic

Let E, F be the double points of the given involution range, let $E\omega$, $F\omega'$ cut at V, where ω, ω' are the circular points at infinity Then $V\omega$, $V\omega'$ are the double lines of the pencil V{A, A′, }, and so this pencil is orthogonal

Similarly, another position is given by the intersection of $E\omega'$, $F\omega$

If the given involution range is elliptic, E and F are conjugate imaginaries, therefore V is real. Q.E.D.

Theorem 117. (1) The projection of a range in involution is a range in involution, the projection of a pencil in involution is a pencil in involution, the double points and double lines project into the double points and double lines in the projected figure

(2) Any involution pencil can be projected into an orthogonal pencil.

(1) follows at once from the fundamental cross ratio theorems.

(2) is effected by projecting the double lines into isotropic lines.

EXERCISE X. d.

1 If VA, VA′ is the perpendicular line pair of an involution, prove that tan AVP tan AVP is constant This corresponds to $mm = k$

2 If V{A, A , B, B , } is an involution pencil with VE, VF as double lines, prove that

(i) $\dfrac{\sin AVP}{\sin A\,VP} \dfrac{\sin AVP'}{\sin A\,VP}$ is constant

(ii) $\sin AVB \ \sin B\,VC \ \sin C'VA = -\sin A'VB' \ \sin BVC' \ \sin CVA$.

(iii) $\sin EVF \ \sin PVP' = 2 \sin EVP \ \sin FVP$

3 Prove that two homographic pencils can be superposed so as to form an involution

4. ABCD is a quadrangle inscribed in a conic S, PH, PK are the tangents to S from any point P on the third diagonal of ABCD, prove that P{H, K, A, C, B, D} is in involution

5 V, V′ are conjugate points w r t a conic, PP′ is a variable chord through V′, prove that VP, VP generate an involution Determine the double lines

6 A, A′, B, B′, C, C′ are the pairs of opposite vertices of a quadrilateral, P is any other point, prove that P{A, A′, B, B′, C, C′} is in involution [Project ∠APA′ and ∠BPB′ into right angles]

7. Prove that conjugate diameters of a conic are line-pairs of an involution Determine the double lines

8. PQR is a self conjugate triangle w r t a system of conics, prove that the tangents from P to any one of the conics generate an involution pencil

9. E is a meet of the diagonals of a quadrilateral circumscribing a conic S; prove that any line through E is cut in involution by S and the sides of ABCD.

10. Prove that any involution range can be projected into a system of point-pairs having a common mid-point.

11. Prove that any transversal is cut in involution by a system of conics (i) through four fixed points, (ii) having a common focus and directrix.

12. A variable rectangular hyperbola passes through two fixed points A, B; a fixed line l perpendicular to AB cuts the curve at P, P′; prove that P, P′ generate an involution on l.

13. Through a point V, lines Vα, Vβ, Vγ are drawn parallel to the sides BC, CA, AB of a triangle; prove that V{A, α; B, β; C, γ} is in involution.

14. PP′ is a variable chord of a given conic; A is a fixed point on the conic. If PP′ passes through a fixed point B, prove that AP, AP′ generate an involution pencil. [Project the conic into a circle with the projection of B as centre.]

15. A, A′; B, B′; C, C′, D, D′ are four fixed point-pairs in involution; H is a variable point on the base; P, Q, R, S are the harmonic conjugates of H w.r.t. the four point-pairs; prove that {PQRS} is constant.

CHAPTER XI

INVOLUTION PROPERTIES OF THE CONIC

Involution Ranges. A, A′, B, B′, C, C′, are a system of pairs of points on a conic, V_1, V_2 are any two other points on the conic. By the fundamental cross-ratio property, if the line pairs V_1A, V_1A', V_1B, V_1B', V_1C, V_1C', form an involution pencil, then also the line pairs V_2A, V_2A', V_2B, V_2B', V_2C, V_2C', form an involution pencil As in Chapter IX, it is therefore unnecessary to specify the particular position of the point V on the conic, when dealing with the cross ratio properties of the pencil V{AA BB′CC′ }

Definition. A system of point pairs A, A′, B, B′, on a conic is called an **involution range of points on the conic** or an **involution range of the second order,** if their joins to any other point on the conic form an involution pencil

It is therefore evident that the necessary and sufficient condition that the point pairs A, A′, B, B, C, C′, on a conic should form an involution is that the ranges {AA BB CC′ }, {A′AB′BC′C } of points on the conic should be homographic

Theorem 118. An involution range of points on a conic exists and is determined uniquely when two point pairs on the conic are given, and there exist two and only two points on the conic (real or conjugate imaginaries), each of which is a coincident point pair of the involution

This follows at once from Theorem 112 (1), (3)

Note The coincident point-pairs are called the **double points of** the involution.

Theorem 119. (1) If A, A′; B, B′; C, C′; ... are point-pairs of an involution range of points on a conic, and if E, F are the double points, then AA′, BB′, CC′, ... concur at the pole O of EF w.r.t. the conic.

(2) If A, A′; B, B′; C. C′; ... are pairs of points on a conic such that AA′, BB′, CC′, ... concur at a point O, then the point-pairs A, A′; B, B′; ... form an involution range on the conic, having as double points the meets E, F of the conic with the polar of O.

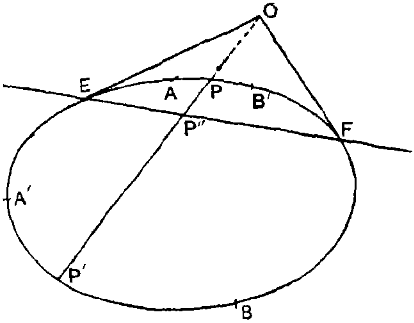

FIG. 92.

(1) By Theorems 112 (3), E{PP′; EF} is harmonic.

∴ E{PP′; OF} is harmonic; similarly, F{PP′; OE} is harmonic. These pencils have a common corresponding ray EF.

∴ the meets of EP, FP; EP′, FP′; EO, FO are collinear.

∴ PP′ passes through O.

(2) Let EF cut PP′ at P″.

Then E{EF; PP′} = E{OP″; PP′} = {OP″; PP′} = −1.

∴ E{EF; PP′} is harmonic.

∴ by Theorem 112 (4), FP, EP′ generate an involution pencil with EE, EF as double lines.

∴ P, P′ generate an involution range of points on the conic with E, F as double points. Q.E.D.

Involution Pencils. There are dual properties corresponding to many of the results in this chapter; as they can be established by the ordinary dual method, we shall omit the proofs and merely enumerate the results. *The reader should draw his own figure in each case.*

Definitions.

(1) A system of tangent pairs a, a'; b, b'; ... to a conic is called an **involution pencil of tangents to the conic** or an **involution pencil of the second order** if their meets with one other tangent (and therefore with any other tangent) to the conic form an involution range.

(2) A tangent to the conic which is a coincident tangent-pair of the involution is called a **double line** of the involution of tangents to the conic.

Theorem 120. An involution pencil of tangents to a conic exists and is determined uniquely when two tangent-pairs to the conic are given; and there exist two and only two tangents to the conic (real or conjugate imaginaries), each of which is a coincident tangent pair of the involution.

This is the dual of Theorem 118.

Theorem 121. (1) If a, a'; b, b'; c, c'; ... are tangent-pairs of an involution pencil of tangents to a conic, and if e, f are the double lines, then aa', bb', cc', ... lie on a straight line, the polar o of ef w.r.t. the conic.

(2) If a, a'; b, b'; c, c'; ... are pairs of tangents to a conic such that aa', bb', cc', ... lie on a straight line o, then the tangent-pairs a, a'; b, b'; ... form an involution pencil of tangents to the conic, having as double lines the tangents e, f to the conic from the pole of o.

This is the dual of Theorem 119.

EXERCISE XI. a.

1. Two chords PQ, RS of a conic meet at H, MN is the tangent at a point M of the conic, prove that MP, MQ, MR, MS, MH, MN form an involution

2 The vertex A of the triangle ABC is a fixed point on a given conic, and the mid point and direction of BC are fixed If AB, AC cut the conic again at B, C, prove that B C passes through a fixed point

3 O is a fixed point, p p' are a variable line pair of a given involution. P, P′ are the feet of the perpendiculars from O to p, p', prove that PP′ passes through a fixed point

4. A variable triangle is inscribed in a given conic, two of its sides pass through fixed points, prove that its vertices trace out homographic ranges of points on the conic

5 A variable quadrilateral circumscribes a given conic, three of its vertices lie on fixed lines, prove that its sides generate homographic pencils of tangents to the conic

6 **If A, A ; B, B ; C, C′ ; are point-pairs of an involution range on a conic, prove that the point at which AA, BB′, CC, concur is the pole w r t the conic of the cross-axis of the homographic ranges {A, B, C, }, {A′, B, C, } of points on the conic**
What is the dual theorem ?

7. A is a fixed point on a given tangent AC to a conic S, P, Q are variable points on AC such that AP, AQ subtend equal angles at a fixed point B, prove that the other tangents from P, Q to S meet on a fixed line

8 **O is a fixed point on a conic S ; PQ is a variable chord of S such that OP, OQ are equally inclined to a fixed line ; prove that PQ passes through a fixed point.**

9 AB is a common tangent of two conics S_1 S_2, from a variable point T on a fixed line, tangents are drawn to S_1 and meet AB at P, Q, find the locus of the meet of the other tangents from P, Q to S_2

10. Tangents are drawn from a variable point on a fixed line to a given conic and meet a fixed tangent to the conic at P, Q, prove that P, Q generate an involution range

11. A is a common point of two conics S_1, S_2, PQ is a variable chord of S_1 passing through a fixed point, AP, AQ meet S_2 at P′, Q′, prove that P′Q′ passes through a fixed point

12. AB is a fixed diameter of a conic S, C is a fixed point on the tangent at B, a variable line through C cuts S at P, Q, BP, BQ meet the tangent at A in P′, Q′, prove that the mid point of P Q′ is fixed

13. A fixed circle S cuts a variable circle of a given coaxal system at P, P′; prove that P, P′ generate an involution range on S.

14. A is a fixed point on a conic S; a variable pair of parallel tangents to S meet a fixed tangent at P, Q; AP, AQ meet S at P′, Q′; find the envelope of P′Q′.

15. A variable circle cuts two fixed circles orthogonally; prove that *the points of intersection generate involution ranges on the fixed circles.*

16. From a variable point on a fixed line, tangents p, q are drawn to a parabola. From a fixed point O, lines OP, OQ are drawn parallel to p, q; prove that OP, OQ generate an involution pencil.

17. Two chords PQ, RS of a conic meet on the chord AB; if O is the pole of AB, prove that AP, AQ; AR, AS; AO, AB form an involution.

18. A, B, C, D, P are five points on a conic; O is any other point; OA, OB cut the conic again at A′, B′; prove that

$$O\{ACBD\} = P\{ACBD\} \times P\{A'CB'D\} \quad \text{or} \quad P\{ACBD\} \div P\{A'CB'D\}$$

19. A fixed circle passes through the centre of a given conic; a variable pair of conjugate diameters of the conic meet the circle at P, Q; prove that PQ passes through a fixed point.

20. A, B, C, P, Q, R are six points on a conic; if R{PABC} = R{QCBA}, prove that PQ meets AC on the tangent at B.

21. T is the pole of a fixed chord PQ of a conic; a variable line cuts TP, TQ at H, K and PQ at a fixed point O; prove that the other tangents from H, K to the conic meet on a fixed line.

Theorem 122. [Desargues' Theorem.]

If a system of conics pass through four fixed points—three of the conics being pairs of lines—then the meets of any line with the conics form point pairs in involution.

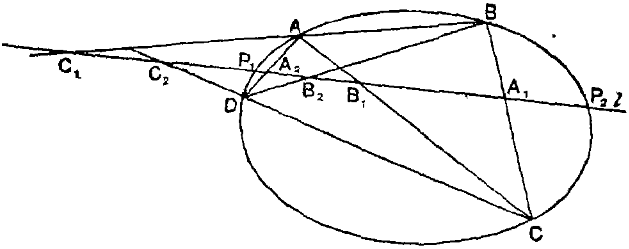

FIG 93

Let A, B, C D be the fixed points and l the given line.

Let the meets of l with BC, AD, CA, BD, AB, CD be A_1, A_2, B_1, B_2; C_1, C_2, and let the meets of l with any conic of the system be P_1, P_2.

Then $$C\{P_1P_2AB\} = D\{P_1P_2AB\},$$

$$\{P_1P_2B_1A_1\} = \{P_1P_2A_2B_2\} = \{P_2P_1B_2A_2\},$$

$\therefore$ A_1, A_2, B_1, B_2, P_1, P_2 form an involution.

Similarly, C_1, C_2, B_1, B_2, P_1, P_2 form an involution

But an involution is defined uniquely by the two point pairs B_1, B_2, P_1, P_2 Therefore A_1, A_2, B_1, B_2, C_1, C_2, P_1, P_2 are in involution, and the meets of l with any other conic of the system are point-pairs of the same involution Q E D

Corollary. If X_1, X_2 is any point pair of the same involution, a conic can be drawn through the six points A, B, C, D, X_1, X_2

Note. This theorem may be proved easily by projection Project B, C into the circular points at infinity, and the result follows from Theorem 109

Definition. A system of conics which pass through four fixed points is called a **pencil of conics.**

The following results are all special cases of Desargues' theorem.

(1) If a line l touches one of a pencil of conics, the point of contact of l is the double point of the involution formed by the meets of l with the conics.

(2) Two and only two conics can be drawn to pass through four points and to touch a given line; and in particular two and only two parabolas can be drawn through four given points.

(3) If PQR is the diagonal point-triangle of the quadrangle ABCD, the meets of any line through P with the pencil of conics through A, B, C, D form an involution having P as one of the double points.

Theorem 123. [Lamé's Theorem]

(1) If two points P, Q are conjugate w.r.t. each of two conics S_1, S_2, then they are conjugate w.r.t. every conic passing through the four common points of S_1, S_2.

(2) The polars of a given point P w.r.t. a pencil of conics are concurrent.

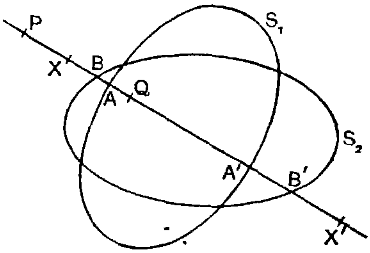

FIG 94.

(1) Let A, A′; B, B′ be the meets of PQ with S_1, S_2.

Since P, Q are conjugate points, {AA′, PQ} and {BB′, PQ} are harmonic. Therefore P, Q are the double points of the involution defined by A, A′; B, B′.

Let X, X′ be the meets of PQ with any conic S of the pencil, determined by S_1, S_2. Then X, X′ is a point-pair of the same involution, ∴ {XX′; PQ} is harmonic.

∴ P, Q are conjugate w.r.t. S.

(2) Since P, Q are conjugate w.r.t. S, the polar of P w.r.t. S passes through Q. Therefore the polar of P w.r.t. any conic of the pencil passes through Q.

∴ the polars are concurrent. Q.E.D.

Theorem 124. [Sturm's Theorem.]

If a system of conics touch four fixed lines—three of these conics being pairs of points—then the tangents from any point L to the conics form line pairs in involution.

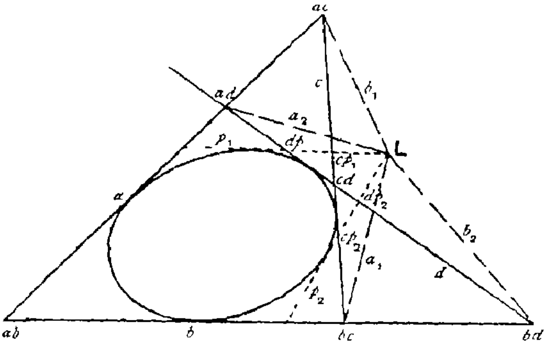

Fig. 95

This is the dual of Desargues' theorem [Th. 122], and is proved by the usual literal and verbal changes.

Corollary. If x_1, x_2 is any line pair of the involution through L, a conic can be drawn to touch the six lines a, b, c, d, x_1, x_2.

Definition. A system of conics which touch four fixed lines is called a **range of conics**.

The following results are all special cases of Sturm's theorem.

(1) If a point L lies on one of a range of conics, the tangent at L is a double line of the involution formed by the line pairs of the tangents from L to the conics.

(2) Two and only two conics can be drawn to touch four lines and to pass through a given point.

(3) If pqr is the diagonal line-triangle of the quadrilateral $abcd$, the tangents from any point on p to the range of conics touching a, b, c, d form an involution having p as one of the double lines.

Theorem 125. (1) If two lines p, q are conjugate w.r.t. each of two conics S_1, S_2, then they are conjugate w.r.t. every conic touching the four common tangents of S_1, S_2.

(2) The poles of a given line p w.r.t. a range of conics are collinear.

(3) The centres of a range of conics are collinear.

(1) and (2) are the duals of Theorem 123, and (3) follows from (2) by taking the given line as the line at infinity.

Desargues' theorem yields a very neat proof of Pascal's property.

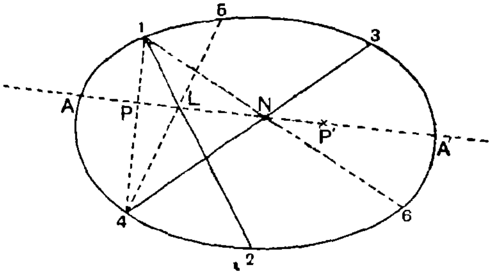

FIG 96

Let 123456 be the inscribed hexagon; let 12, 45 meet at L and 34, 61 at N, join LN and produce it to meet the conic at A, A′; let LN meet 14 at P, and take a point P′ on LN so that the point-pairs L, N, A, A′, P P′ form an involution

Apply Desargues' theorem to the inscribed quadrangle 1234 then P′ lies on 23.

Apply Desargues' theorem to the inscribed quadrangle 1654; then P′ lies on 56.

∴ 23, 56 meet at P′, which is collinear with L, N. Q.E.D.

Construction. To construct a conic to pass through three given points and to touch two given lines

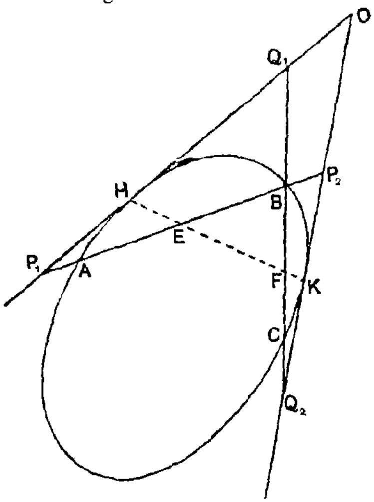

FIG 97

Let A, B, C be the given points and OH, OK the given lines It is required to find the points H, K at which the conic touches OH, OK

Let AB cut OH, OK at P_1, P_2 and HK at E, and let BC cut OH, OK at Q_1, Q_2 and HK at F

By Desargues theorem, E is a double point of the involution determined by A, B, P_1, P_2, and therefore has one of two possible positions which can be constructed

Similarly, F is a double point of the involution determined by B, C, Q_1, Q_2, and therefore has one of two possible positions which can be constructed

The line EF which meets OH, OK at the required points H, K has therefore four possible positions, and so four conics can be constructed to satisfy the given conditions

EXERCISE XI. b.

1. Deduce from Desargues' theorem that the intercept of any tangent to a hyperbola between the asymptotes is bisected at the point of contact

2. Two conics touch the same line at P, Q respectively and cut at A, B, C, D ; R is the mid-point of PQ ; prove that the conic through A, B, C, D, R has one asymptote parallel to PQ

3 Two conics cut at A, B, C, D , a straight line touches them at P, Q and cuts AC, BD at H, K , prove that {PQ , HK} is harmonic

4. O is the mid-point of a chord AB of a conic , C, D are points on AB equidistant from O , two lines CPQ, DRS cut the conic at P, Q and R, S , if PR, QS meet AB at X, Y, prove that $CX = DY$.

5 P is any point on a chord AB of a conic , a line through P cuts the conic in C, D and the tangents at A, B to the conic in Q, R , prove that $PC\ PR\ QD = PD\ PQ\ CR$

6 H is the mid point of a chord AB of a conic , PQ, RS are two chords through H , PR, QS meet AB at X, Y , prove that $AX - BY$

7 A pair of common chords of two conics S_1, S_2 meet at T , the tangent TP to S_1 cuts S_2 at H, K , prove that {TP , HK} is harmonic

8. A line drawn through a point P on a hyperbola, parallel to an asymptote, cuts two pairs of opposite sides of an inscribed quadrangle in H, H′ , K, K′ , prove that $PH\ PH' = PK\ PK'$

9 The three pairs of opposite sides of a quadrangle inscribed in a hyperbola meet an asymptote in P, P′ , Q, Q′ , R, R′ , prove that $PQ = P'Q'$ and $QR = Q'R'$

10 A diameter of a parabola meets a chord PQ in H, the tangents at P, Q in M, N and the curve in O ; prove that $OH^2 = OM\ ON$

11. PQ is a chord of a conic bisecting another chord AB at O , the tangents at P, Q meet AB in S, T , prove that $AS = BT$

12 P, Q, R, S, T are five points on a conic , PQ, RS, PR, QS cut the tangent at T in H, K, M, N , prove that $\frac{1}{TH} + \frac{1}{TK} = \frac{1}{TM} + \frac{1}{TN}$

13 From a fixed point, lines are drawn parallel to the sides of a quadrangle , prove that they form an involution

14. A variable conic passes through four fixed points A, B, C, D and cuts a fixed conic through A, B in P, Q , prove that PQ passes through a fixed point

15. The circle of curvature at a point P of a conic cuts the conic again at Q , prove that PQ and the tangent at P divide harmonically the other common tangent of the circle and the conic

16 The tangents at the points P, P′ on a hyperbola meet an asymptote at Q, Q′ , prove that PP′ bisects QQ′

17. A variable chord PQ of a conic passes through a fixed point A. B is another fixed point; BP, BQ meet the conic again at R, S; prove that RS passes through a fixed point.

18. ABC is a given triangle inscribed in a conic; O is a fixed point on the conic; a variable line through O cuts the conic again at P and BC, CA, AB at Q, R, S; prove that {PQRS} is constant.

19. From a fixed point O, lines OA′, OB′, OC′ are drawn parallel to the sides BC, CA, AB of a triangle; prove that OA, OA′; OB, OB′; OC, OC′ form an involution.

20. PCP′, DCD′ are conjugate diameters of an ellipse; the lines joining D, D′ to a variable point Q on the ellipse meet the tangent at P in X, Y; prove that $\frac{1}{PX} \sim \frac{1}{PY} = \frac{1}{CD}$.

21. Two parabolas touch at P and cut at Q, R; prove that PQ, PR are harmonically conjugate to the diameters through P.

22. A system of conics circumscribe a given triangle and have a common pair of conjugate points; prove that they pass through a fourth fixed point.

23. A transversal cuts the sides BC, CA, AB of a triangle at P, Q, R; three other points P′, Q′, R′ are taken such that P, P′; Q, Q′; R, R′ form an involution; prove that AP′, BQ′, CR′ are concurrent.

24. The sides BC, CA, AB of a triangle touch a conic at D, E, F; PT is the tangent at any point P of the conic; prove that

$$P\{TABC\} = P\{TDEF\}.$$

25. Three equidistant lines, parallel to an asymptote of a conic, meet the curve at P, Q, R; prove that any line parallel to the other asymptote is cut harmonically by the curve and PQ, QR, RP.

26. AB, CD are conjugate chords of a conic; any chord AP meets BC, CD, DB at Q, R, S; prove that {PR, QS} is harmonic.

27. O is one of the common points of two conics and OP, OQ are the tangents at O; if A, A′ is one pair of opposite vertices of the quadrilateral formed by their common tangents, prove that O{AA′; PQ} is harmonic.

28. ABCD is a quadrilateral circumscribing a conic; a tangent at any point P on the conic cuts CD at Q; AP, BP cut CD at L, M; prove that {QMLC} = {QDCL}.

29. ABCD, A′B′C′D′ are two quadrilaterals circumscribing a conic; if AA′, BB′, CC′ concur at a point O, prove that DD′ also passes through O.

30. A, A′; B, B′; C, C′ are the pairs of opposite vertices of a quadrilateral; {AA′; PQ} and {BB′; RS} are harmonic ranges; prove that PR and QS cut CC′ harmonically.

31. Prove that the feet of the perpendiculars from the six vertices of a quadrilateral to any straight line form an involution.

32. Four circles have one common point; prove that their radical axes form an involution pencil.

33. A, A′ ; B, B′ are two pairs of opposite vertices of the quadrilateral formed by the common tangents of two conics which cut orthogonally at P ; prove that $\angle APB = \angle A'PB'$.

34. ABC is a triangle circumscribing a conic ; the polar of A meets BC at D ; a tangent at any point Q of the conic meets the other tangent from D in T ; prove that T{BC , QA} is harmonic.

35. The poles of two fixed lines w r t a variable conic are fixed points ; prove that the locus of the centre of the conic is a straight line.

36. Construct a conic to pass through three given points and have double contact with a given conic.

37. Construct a conic to pass through two given points and touch three given lines.

38. Construct a conic, given four points on it and a pair of points conjugate w r t it.

Theorem 126. [Frégier's Theorem.]

If a variable chord PP′ of a conic subtends a right angle at a fixed point V on the conic, then it passes through a fixed point F situated on the normal at V.

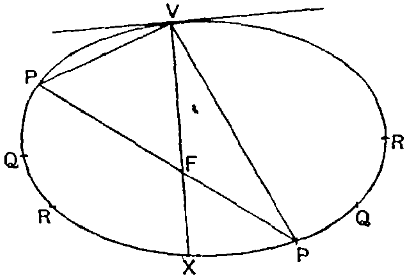

FIG 98

Let QQ′, RR′, SS′, … be other positions of the chord.

Since $\angle PVP' = 90° = \angle QVQ' = \ldots$, the line pairs VP, VP′ ; VQ, VQ′ ; … form an involution.

∴ P, P′ ; Q, Q′ ; R, R′ ; … form an involution range of points on the conic ; therefore PP′, QQ′, RR′, … are concurrent.

But one position of PP′ is the normal VX at V, since $\angle VVX = 90°$. Therefore PP′ passes through a fixed point F on the normal at V.

Q E D.

Definition. The fixed point F in Theorem 126 is called the **Frégier point** of the point V.

EXERCISE XI. c.

1. If V is a point on a rectangular hyperbola, prove that the Frégier point of V is at infinity.

2. QR is a chord of a rectangular hyperbola parallel to the normal at a point P on the curve. prove that $\angle QPR = 90^\circ$.

3. Prove Frégier's theorem by reciprocating w.r.t V.

4. F is the Frégier point of a point V on a parabola; prove that VF is bisected by the axis of the parabola

5. F is the Frégier point of a point V on a central conic; ACA', BCB' are the principal axes: prove that CA, CB are the bisectors of $\angle$VCF.

6. On a chord PQ of a rectangular hyperbola as diameter a circle is described, cutting the curve again at V, V'; prove that the normals at V, V' are parallel.

7. If a chord of a parabola subtends a right angle at the vertex A and meets the axis at F, prove that AF is equal to the latus rectum.

8. Prove that circles whose diameters are parallel chords of a rectangular hyperbola are coaxal.

9. A parabola, focus S, reciprocated w r.t. a point C, becomes a conic σ; prove that S becomes the polar of the Frégier point of C w r.t. σ.

Theorem 127. (1) Every conic through the four common points A, B, C, D of two rectangular hyperbolas is a rectangular hyperbola; and D is the orthocentre of the triangle ABC.

(2) If a rectangular hyperbola circumscribes a triangle ABC, it passes through its orthocentre.

(1) Let ω, ω' be the circular points at infinity; then ω, ω' are conjugate points w.r.t. the two given rectangular hyperbolas through A, B, C, D, and therefore by Theorem 123 ω, ω' are conjugate points w.r.t. every conic through A, B, C, D.

$\therefore$ every such conic is a rectangular hyperbola.

But AB, CD; AC, BD; AD, BC are three conics of the pencil; therefore they are perpendicular line-pairs.

$\therefore$ D is the orthocentre of the triangle ABC.

(2) Let the perpendicular from A to BC cut the curve in D; then the given rectangular hyperbola and the perpendicular line-pair AD, BC determine a pencil of rectangular hyperbolas.

But AB, CD is a conic of this pencil, and so AB is perpendicular to CD.

$\therefore$ D is the orthocentre of the triangle ABC. Q.E.D.

Note. This theorem was published in 1821 by Brianchon and Poncelet in a joint memoir. It is capable of very simple analytical proof.

EXERCISE XI. d.

1. **PQR is a triangle, right-angled at P, inscribed in a rectangular hyperbola; prove that the tangent at P is perpendicular to QR**

2. **PQ is a chord of a rectangular hyperbola; the circle on PQ as diameter cuts the curve again at R, S; prove that RS is a diameter of the hyperbola**

3. (Steiner's Theorem) **Prove that the orthocentre of a triangle circumscribing a parabola lies on the directrix** [Reciprocate Theorem 127]

4. PQR is a self conjugate triangle w r t a conic S_1, a conic S_2 inscribed in PQR, touches a directrix of S_1, prove that the director circle of S_2 passes through a focus of S_1

5. Prove that the pedal triangle of a triangle inscribed in a rectangular hyperbola is self conjugate for the hyperbola

6. If the normal at a point P on a rectangular hyperbola meets the curve again at Q prove that the radius of curvature at P equals $\frac{1}{2}$PQ

7 If the normal at a point P on a rectangular hyperbola, centre C, meets the curve again at Q, and if R is the mid point of PQ, prove that $\angle RCP = 90°$

8 ABCD is a parallelogram inscribed in a rectangular hyperbola, PH, PK, PL, PM are the perpendiculars from a point P on the curve to AB, BC, CD DA prove that PH PL PK PM

9. The mid points of the sides of a variable triangle move on a rectangular hyperbola, find the locus of its circumcentre

10. HK is a variable chord of a rectangular hyperbola, fixed in direction, PP′ is the diameter perpendicular to HK, find the locus of the meet of HP, KP

Theorem 128. (1) Through any given point T, two conics can be drawn belonging to a given confocal system; and these conics cut each other orthogonally.

(2) TX, TY are the tangents to the two confocals through T; if S, H are the real foci, and if TP, TQ are the tangents from T to any conic of the system, then TX, TY are the internal and external bisectors of the angles STH and PTQ.

(3) *TX, TY are conjugate lines w.r.t. each conic of the system.*

(4) The locus of the poles of a given line w.r.t. a system of confocal conics is a line perpendicular to the given line.

(1) Let S, H, S′, H′ be the foci of the given system and ω, ω' the circular points at infinity.

Then S, H, S′, H′, ω, ω' are the pairs of opposite vertices of the quadrilateral circumscribing each conic of the confocal system.

Let TX, TY be the double lines of the involution defined by TS, TH, Tω, Tω'. Then, by Theorem 124, Corollary, a conic of the system can be drawn to touch TX at T, and similarly a second conic of the system to touch TY at T.

Since TX, TY are double lines, T{XY, $\omega\omega'$} is harmonic, and therefore $\angle$XTY$=90°$.

∴ the conics cut orthogonally at T.

(2) By Theorem 124, TP, TQ are a line pair of the same involution; since the double lines are at right angles, they are the bisectors of the angles between each line pair of the pencil.

∴ TX, TY are the bisectors of angles STH, PTQ.

(3) Since TX, TY are harmonically conjugate to each pair of tangents from T to the system of conics, they are conjugate lines w.r.t. each conic.

(4) Let a conic of the system touch the given line TX at T, and let TY be the tangent at T to the second confocal through T.

By (1), TX is perpendicular to TY.

By (3), TY is conjugate to TX w.r.t. each conic of the system, and therefore the locus of the poles of TX w.r.t. the system is the perpendicular line TY. Q.E.D.

Theorem 129. [Plücker's Theorem.]

(1) The three circles whose diameters are the joins of opposite vertices of a quadrilateral are coaxal; and the director circles of all conics inscribed in the quadrilateral belong to the same coaxal system.

(2) The centres of all conics inscribed in a given quadrilateral lie on a line passing through the mid-points of the joins of opposite vertices of the quadrilateral.

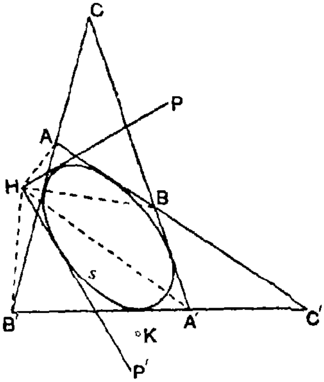

FIG. 99.

(1) Let A, A′; B, B′; C, C′ be the pairs of opposite vertices of the quadrilateral. Let the circles on AA′, BB′ as diameters meet at H, K. Let HP, HP′ be the tangents from H to any conic S inscribed in the quadrilateral.

By Theorem 124, HA, HA′; HB, HB′; HC, HC′; HP, HP′ form an involution.

But $\angle AHA' = 90° = \angle BHB'$, angles in semicircle.

∴ since two line-pairs are at right angles, every line-pair of the involution is at right angles.

∴ the circle on CC′ as diameter passes through H; and the director circle of S passes through H. Similarly, the circle on CC′ as diameter and the director circle of S pass through K.

∴ the circles on AA′, BB′, CC′ as diameters are coaxal, and the director circles of all conics inscribed in the quadrilateral belong to the same coaxal system.

(2) The centres of the circles of a coaxal system are collinear, and the director circle of a conic is concentric with the conic.

Therefore the centres of all conics inscribed in the quadrilateral lie on a straight line which passes through the mid-points of AA′, BB′, CC′. Q.E.D.

Note. Theorem 129 (2) is due to Newton. Theorem 129 (1) was discovered independently by Gaskin; for an alternative method of proof, see No. 10 below.

Theorem 137. (1) E, F, G, H are the feet of the four normals from a point O to a conic, centre C; M, N are the feet of the perpendiculars from the pole T of EF to the axes ACA′, BCB′; if GH cuts CA, CB at M′, N′, then M′C = CM and N′C = CN.

(2) [*Joachimsthal's Theorem.*] *The circle drawn through the feet of three of the normals from a point to a conic cuts the conic again at the point which is diametrically opposite to the foot of the fourth normal.*

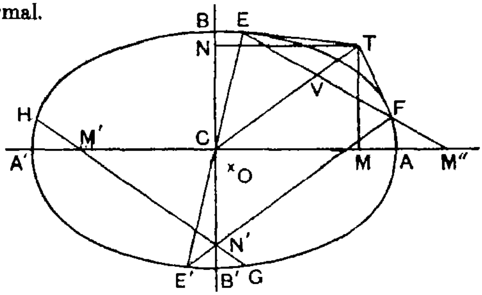

FIG. 100

(1) Let EF meet CA at M″.

By Theorem 101, the points E, F, G, H lie on a rectangular hyperbola through C and having its asymptotes parallel to CA, CB; Now the transversal AA′ is cut in involution by all conics through EFGH; and point-pairs of this involution are M′, M″; A, A′; C, ∞. therefore C is the centre of this involution.

$$\therefore \ CM' \,.\, CM'' = CA \,.\, CA' = -CA^2.$$

Since T is the pole of EF, TM is the polar of M″, and therefore {AA′; MM″} is harmonic.

$\therefore$ CM . CM″ = CA²; $\therefore$ CM = −CM′ = M′C, and similarly CN = N′C.

(2) Let EC meet the conic again at E′ and CT meet EF at V.

Since EV = VF and EC = CE′, CVT is parallel to E′F.

But CT, MN are equally inclined to CA, being the diagonals of a rectangle: and by (1), MN is parallel to M′N′ or GH; also CT is parallel to E′F.

$\therefore$ GH and E′F are equally inclined to CA, an axis of the conic.

$\therefore$ the circle through FGH cuts the conic again at E′. [See Exercise VI. d, No. 16, p. 107.] Q.E.D.

EXERCISE XI. e.

1. From a point A on a conic S_1, tangents AB, AC are drawn to a confocal conic S_2, prove that AB, AC are equally inclined to the tangent at A to S_1

2 Given two tangents to a conic and one focus, find the locus of the other focus

3. Y, Z are the feet of the perpendiculars from a focus S of an ellipse to two tangents TY, TZ, prove that the perpendicular from T to YZ passes through the other focus

4 S, H are the foci of an ellipse, T is the pole of a chord PSQ, prove that the normals at P, Q meet on HT

5 ABCD is a parallelogram circumscribing a conic, focus S, prove that the circles ABS, ADS are equal

6. ABCD is a fixed parallelogram circumscribing a variable conic σ, prove that the locus of the foci of σ is a rectangular hyperbola through A, B, C, D

7 One focus of an ellipse inscribed in a triangle is at the orthocentre, prove that the centre of the ellipse is at the nine point centre

8. PQ is a diameter of a conic, R is a point on the curve such that PR, QR make equal angles with the tangent at R, prove that the pole of PR lies on the director circle

9 **The sides BC, CA, AB of a triangle ABC, inscribed in a conic S_1, touch a confocal conic S_2 at P, Q, R, prove that the escribed circles of ABC touch the sides at P, Q, R.**

[Let the tangents at A, B, C to S_1 form the triangle XYZ, and prove that X, Y, Z are the ex centres, Theorem 128 (2), (4)]

10. If H lies on the director circle of each of two conics, prove, by reciprocating w r.t. H, that it lies on the director circle of every conic touching the four common tangents of the two given conics

11 If a system of conics touches four fixed lines prove that the radical axis of their director circles is the directrix of the parabola touching the four lines

12 T is the pole of a chord PQ of a conic σ, prove that T is a limiting point of the coaxal system determined by the director circle of σ and the circle on PQ as diameter

13. A, B are the centres of the two rectangular hyperbolas which can be inscribed in a given quadrilateral, prove that any circle through A, B is orthogonal to the director circle of any conic inscribed in the quadrilateral

14. A variable conic is inscribed in a given triangle, if its director circle passes through one fixed point, prove that it must also pass through a second fixed point

15. By taking A′B′ as the line at infinity in Theorem 129 (1), deduce that the orthocentre of a triangle circumscribing a parabola lies on the directrix.

16. Prove that the lines bisecting the joins of opposite vertices of each of the five quadrilaterals formed by five straight lines are concurrent.

17. If a parabola is inscribed in the cyclic quadrilateral ABCD, prove that AC, BD meet on its directrix

18. Prove that the circumcircle of a triangle self-conjugate w.r.t. a conic is orthogonal to the director circle of the conic.

19. Prove that the centre of a conic is the radical centre of the circum-circles of all triangles self-conjugate w r t the conic

20. ABCD is a quadrilateral circumscribing a parabola . prove that the join of the mid points of AC, BD is parallel to the axis of the parabola.

21. A parabola is drawn to touch the four common tangents of two rectangular hyperbolas , prove that its directrix is the perpendicular bisector of the join of the centres of the hyperbolas

22. T is the pole of the chord PQ of a parabola , prove that the directrix of the parabola is halfway between T and the polar of T w r t the circle on PQ as diameter

23. Four conics circumscribe the triangle ABC and have a common focus D , prove that the director circle of any conic, touching the four corresponding directrices, passes through D

24. Prove that the circumcircle of the diagonal line triangle of a quadrilateral cuts the line passing through the mid points of the joins of opposite vertices at the centres of the rectangular hyperbolas which are inscribed in the quadrilateral

25. [Harvey's Theorem] **If P, Q, R are the feet of the normals to a parabola from any point, prove that the circumcircle of the triangle PQR passes through the vertex of the parabola.**

26. P is a fixed point on a central conic , Q is a variable point on the normal at P ; L, M, N are the feet of the other normals from Q to the conic ; prove that the sides of the triangle LMN envelope a parabola.

27. P, Q, R, S are the feet of the normals from a variable point to a conic , PQ passes through a fixed point, prove that RS touches a fixed conic

28. PQ is a chord of a parabola fixed in direction ; prove that the normals at P, Q meet on a fixed line

29. T is the pole of a chord PQ of a conic S_1 TR is a tangent at R to a confocal conic S_2, then PR, QR touch a conic confocal with S_1 and S_2.

30. Two points S and H are the foci of a variable conic inscribed in a given triangle Prove that, if S describes a straight line, H describes a conic circumscribing the triangle

Theorem 131. (1) P is a variable point on a fixed line l, P$'$ is the point which is conjugate to P w r t. the pencil of conics through four fixed points A, B, C, D, then the locus of P$'$ is a conic σ, which passes through the poles of l w r t the conics of the pencil

(2) The conic σ passes through the following eleven points

If l cuts AB at H and if H$'$ is the harmonic conjugate of H w r t A, B, then σ passes through H$'$ and the corresponding five points on the other five sides of the quadrangle ABCD, σ also passes through the three vertices of the diagonal point triangle of ABCD and through the two double points of the involution formed by the meets of l with the pencil of conics

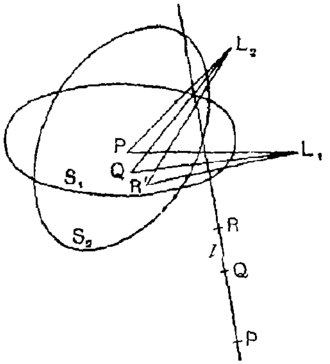

FIG 101

(1) Let L_1, L_2, be the poles of l w r t S_1 S_2, of the pencil, P, Q, R, . are a system of points on l, P$'$, Q$'$, R$'$, are their conjugates w r t two (and therefore all) conics of the pencil

∴ $P'L_1$, $Q'L_1$, $R'L_1$, . and $P'L_2$, $Q'L_2$, $R'L_2$, are the polars of P, Q, R, w r.t S_1, S_2

$$\therefore L_1\{P', Q', R', \ \} = \{P, Q, R, \ \} = L_2\{P', Q', R', \ \}$$

Therefore the locus of P$'$ is a conic σ through L_1, L_2, and similarly through the other poles L_3, L_4, . of l w.r t. the other conics of the pencil.

(2) Since {AB , HH′} is harmonic, H, H′ are conjugate points w r t. every conic of the pencil , but H lies on l , ∴ σ passes through H . Similarly, σ passes through the corresponding five points on AC, AD, BC, BD, CD

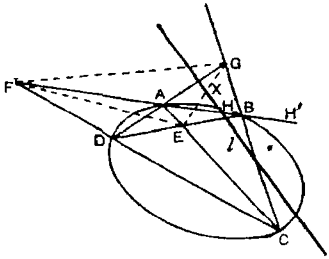

FIG 102

Let EFG be the diagonal point triangle of ABCD, and let X be the meet of l and EG Then, since EG is the polar of F, F and X are conjugate points w r t every conic of the pencil , but X lies on l , ∴ σ passes through F Similarly, σ passes through E and G.

Lastly, if λ, μ are the double points of the involution on l, λ and μ are conjugate points w r t each conic of the pencil, and therefore σ passes through λ, μ Q E D.

Note. The conic locus σ of Theorem 131 is called the **eleven-point conic** of the pencil of conics through A, B, C, D corresponding to the line l [This name is due to Dr Taylor]

We may state Theorem 131 as follows

The locus of the poles of a given line l w r t a pencil of conics is the eleven point conic corresponding to l

Theorem 132. The locus of the centres of a pencil of conics circumscribing a fixed quadrangle is a conic circumscribing the common self conjugate triangle of the pencil and passing through the mid points of the six sides of the quadrangle, and having its asymptotes parallel to the axes of the two parabolas of the pencil

This is the special case of Theorem 131 which arises when l becomes the line at infinity.

Theorem 133. (1) p is a variable line through a fixed point L, p' is a line which is conjugate to p w.r.t. a range of conics touching four fixed lines a, b, c, d, then the envelope of p' is a conic σ, which touches the polars of L w.r.t. the conics of the range

(2) The conic σ touches the following eleven lines:

If h is the join of L and ab, and if h' is the harmonic conjugate of h w.r.t. a, b, then σ touches h' and the corresponding five lines through the other five vertices of the quadrilateral $abcd$, σ also touches the three sides of the diagonal line triangle of $abcd$ and the two double lines of the involution formed by the tangents from L to the range of conics

This is merely the dual of Theorem 131

Note. The conic envelope σ of Theoren 133 is called the **eleven-line conic** of the range of conics touching a, b, c, d corresponding to the point L.

We may state Theorem 133 as follows

The envelope of the polars of a given point L w.r.t. a range of conics is the eleven line conic corresponding to L.

EXERCISE XI. f.

1. Prove that the centre of a rectangular hyperbola circumscribing a given triangle lies on the nine-point circle of the triangle

2. ABC is a triangle inscribed in a rectangular hyperbola, centre O, H is its orthocentre, HO meets the hyperbola at D, prove that A, B, C, D are concyclic

3. If two conics of a four point pencil have parallel axes, prove that all the conics have parallel axes and that one of them is a circle

4. The conic σ is the locus of the centres of all conics of a four point pencil, P is any point on σ, prove that the polars of P w.r.t. conics of the pencil are parallel

5. Find the locus of the centre of a variable rectangular hyperbola (i) if it touches a fixed line at a fixed point and passes through another fixed point, (ii) if it passes through a fixed point A and if its circle of curvature at A is fixed

6. If σ is the conic-locus of the centres of all conics through four fixed points A, B, C, D, prove that the centre of σ is the mean centre of A, B, C, D.

7. A variable conic circumscribes a fixed cyclic quadrangle ABCD, prove that the locus of its centre is a rectangular hyperbola, and that the axes of the two parabolas through A, B, C, D intersect at right angles.

8. Prove that the nine point circles of the triangles ABC, ACD, ABD, BCD have one common point

9. A, B, C, D are four points on a hyperbola, prove that the asymptotes of the hyperbola are parallel to a pair of conjugate diameters of the centre locus of all conics through A B, C, D

10. Five quadrangles are formed from five points, no three of which are collinear, prove that the five conics which pass through the mid-points of the sides of the quadrangles have one common point.

11. PQ is a diameter of a rectangular hyperbola, any circle through P, Q cuts the hyperbola again at R, S, prove that RS is a diameter of the circle

12. A circle meets a rectangular hyperbola at P, Q, R S, if PP′ is a diameter of the hyperbola, prove that P′ is the orthocentre of QRS

13. A rectangular hyperbola circumscribes an equilateral triangle ABC, prove that its centre lies on the incircle of ABC

14 If a conic passes through the in centre and ex centres of a triangle, prove that its centre lies on the circumcircle of the triangle

15. A variable conic, centre P, passes through four fixed points, Q is a point on the conic, the tangent at which has a fixed direction, prove that PQ passes through a fixed point

16. EFG is the diagonal point triangle of the quadrangle ABCD, P is any point, EQ, FR, GS are the harmonic conjugates of EP, FP, GP w r t the sides of the quadrangle which meet at E, F, G respectively, prove that EQ, FR, GS are concurrent

17 Two conics circumscribe a triangle and touch at one of the angular points; prove that their centres, their point of contact and the mid-points of the sides lie on a conic

CHAPTER XII

MISCELLANEOUS PROPERTIES

Theorem 184. (1) If two triangles circumscribe a conic, their six vertices lie on another conic.

(2) If two triangles are inscribed in a conic, their six sides touch a conic.

(3) The circumcircle of a triangle formed by three tangents to a parabola passes through the focus of the parabola.

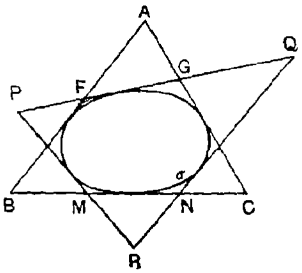

FIG 103

(1) Let ABC, PQR be two triangles circumscribing the conic σ.

Let AB, AC cut PQ in F, G, and let RP, RQ cut BC in M, N.

Then $A\{PBCQ\} = A\{PFGQ\} = \{PFGQ\}$

$= \{MBCN\}$, Chasles' Theorem,

$= R\{MBCN\} = R\{PBCQ\}$.

$\therefore$ A, R, P, B, C, Q lie on a conic.

(2) This is the dual of (1).

(3) Suppose in Fig 102, Q and R are the circular points at infinity, then the conic is a parabola with P as focus, and the conic through A, B, C, P, Q, R becomes a circle through A, B, C, P. Q.E.D

Corollary. If two conics σ, Σ are such that there exists one triangle which is inscribed in Σ and circumscribed about σ, then an unlimited number of such triangles exist

Suppose ABC is the given triangle, let *any* tangent to σ cut Σ in P, Q, and let the other tangents from P, Q to σ meet at R

Then, by Theorem 134 (1), A, B, C, P, Q, R lie on a conic, but A, B, C, P, Q lie on Σ, and a conic is fixed by five points, therefore R lies on Σ

EXERCISE XII. a.

1 The in-centre of a triangle inscribed in a rectangular hyperbola lies on the curve, prove that the in circle passes through the centre of the hyperbola

2 A, P are the poles of the chords BC, QR of a conic, prove that A, B, C, P, Q, R lie on a conic

3 A variable parabola has a fixed self conjugate triangle, prove that the locus of its focus is the nine point circle of the triangle

4 Through each of two points P, Q a pair of lines is drawn so as to form a cyclic quadrilateral ABCD, prove that the focus of the parabola inscribed in ABCD lies on PQ.

5 If two circles S_1, S_2 radii r_1, r_2, centres A_1, A_2 are such that triangles can be inscribed in S_1 and circumscribed to S_2, prove that $-2r_1r_2$ is equal to the power of A_2 w r t S_1

6 F is the focus of an ellipse S_1, S_2 is a circle, centre F, of radius equal to the major axis, prove that an unlimited number of triangles can be inscribed in S_2 and circumscribed about S_1, and that the ortho centre of every such triangle is at the second focus of S_1

7 If two conics Σ_1, Σ_2 are such that one quadrilateral ABCD can be inscribed in Σ_1 and circumscribed about Σ_2, prove that an unlimited number of such quadrilaterals exist

[Project Σ_1 into a circle having the projection of the meet of AC, BD as centre]

8 If ABCD is a variable quadrilateral inscribed in one fixed conic and circumscribed about another fixed conic, prove that AC cuts BD at a fixed point

9. T is the pole of a chord PQ of a parabola, focus S; prove that TQ touches the circle through T, P, S.

10. T is the pole of a chord PQ of a hyperbola S_1, centre C; a hyperbola S_2 is drawn through T, P, Q with asymptotes parallel to those of S_1; prove that S_2 passes through C and that its centre lies on CT.

11. A conic is inscribed in a triangle ABC and passes through the circumcentre of ABC; prove that the circumcircle touches the director circle of the conic.

12. T is the pole of a chord PQ of a parabola, focus S; O is the circumcentre of TPQ, prove that $\angle OST = 90°$.

Theorem 135. *If two triangles are self-conjugate w.r.t. a conic, then their six sides touch a conic and their six vertices lie on a conic.*

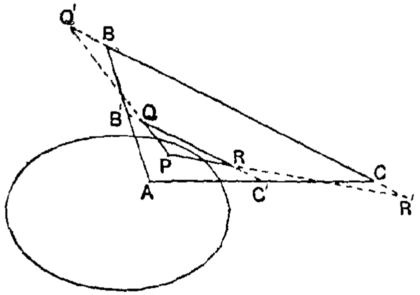

FIG 104

Let ABC, PQR be the two triangles, let PQ, PR meet BC in Q′, R′, and let AB, AC meet QR in B′, C′.

Now Q′ is the meet of PQ, BC, whose poles are R, A

∴ the polar of Q′ is AR, and similarly the polar of R′ is AQ

∴ {BQ′R′C} = A{CRQB}, pencil of polars,
= {C′RQB′} = {B′QRC′}.

∴ BB′, QQ′, RR′, CC′ touch a conic, touching BC, B′C′.

∴ the sides of ABC, PQR touch a conic.

∴ by Theorem 134, the six vertices lie on a conic Q.E.D.

Theorem 136. (1) If two triangles are inscribed in the same conic, there exists a conic w r t which both triangles are self conjugate.

(2) If two triangles circumscribe the same conic, there exists a conic w r t. which both triangles are self conjugate.

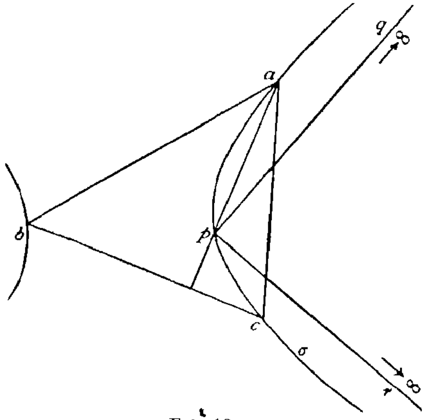

FIG 105

(1) Let ABC, PQR be the two triangles inscribed in a conic Σ

Project QR to infinity and P into the orthocentre of the projection of the triangle ABC Denote corresponding elements of the new figure by small letters

Since σ circumscribes abc and passes through its orthocentre p, σ is a rectangular hyperbola But q, r are points at infinity on σ,

$\therefore \angle qpr = 90$

$\therefore$ pqr is a self conjugate triangle w r t any circle, centre p

Let σ_1 be the circle w r t which the triangle abc is self conjugate, then the centre of σ_1 is at the orthocentre of abc Therefore abc and pqr are both self conjugate w r t σ_1

$\therefore$ in the original figure, there exists a conic Σ_1 w r t. which ABC and PQR are each self conjugate

(2) If two triangles circumscribe the same conic, their six vertices lie on another conic, and therefore the required result follows from (1).

Q.E.D

Theorem 137. (1) If two conics S_1, S_2 are such that there exists one triangle which is inscribed in S_1 and is self-conjugate to S_2, then an unlimited number of such triangles exist.

(2) If two conics S_1, S_2 are such that there exists one triangle which is circumscribed about S_1 and is self-conjugate to S_2, then an unlimited number of such triangles exist.

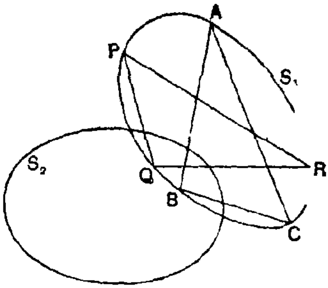

FIG. 106.

(1) Let ABC be inscribed in S_1 and self-conjugate to S_2.

Take any point P on S_1 and let the polar of P w.r.t. S_2 cut S_1 at Q; let the line through P conjugate to PQ w.r.t. S_2 cut the polar of P at R.

Then PQR is self-conjugate w.r.t. S_2.

$\therefore$ by Theorem 135, A, B, C, P, Q, R lie on a conic.

But A, B, C, P, Q lie on S_1, and a conic is determined by five points; therefore R lies on S_1.

(2) This follows by a method similar to that used for (1), or it may be treated as the dual of (1). Q.E.D.

Definitions.

(1) The conic S_1 is said to be **harmonically circumscribed** to the conic S_2 if S_1 circumscribes one triangle (and therefore an unlimited number) *self-conjugate* w.r.t. S_2.

(2) The conic S_1 is said to be **harmonically inscribed** in the conic S_2 if S_1 is inscribed in one triangle (and therefore in an unlimited number) self-conjugate w.r.t. S_2.

Theorem 188. If a conic S_1 is harmonically circumscribed to a conic S_2, then S_2 is harmonically inscribed in S_1.

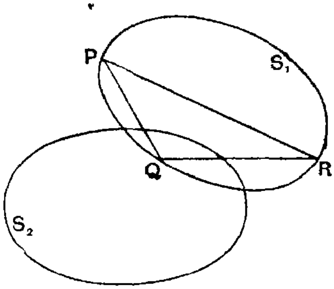

FIG. 107.

PQR is a triangle inscribed in S_1 and self-conjugate w.r.t. S_2.

Let S be a conic w.r.t. which S_1 and S_2 are reciprocal. Reciprocate the figure w.r.t. S and consider in one system the new figure and the old figure.

S_1 becomes S_2, and S_2 becomes S_1; PQR becomes a triangle *pqr* circumscribing S_2 and self-conjugate w.r.t. S_1.

$\therefore$ S_2 is harmonically inscribed in S_1. Q.E.D.

Corollary. If PQR is a triangle circumscribing a rectangular hyperbola S, the polar circle σ of PQR passes through the centre C of S.

Denote the circular points at infinity by ω, ω'.

By hypothesis S is harmonically inscribed in σ; therefore σ is harmonically circumscribed about S.

But $C\omega\omega'$ is a triangle self-conjugate w.r.t. S, and two of its vertices ω, ω' lie on σ; therefore C also lies on σ.

Theorem 139. If a triangle is inscribed in a conic S_1 and circumscribed about a conic S_2, it is self-conjugate w.r.t. a conic w.r.t. which S_1 and S_2 are reciprocal.

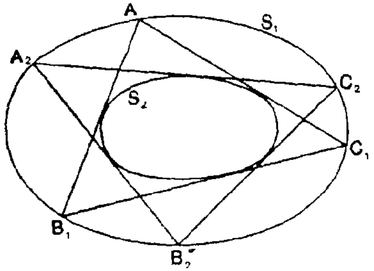

FIG 108

Take two triangles $A_1B_1C_1$, $A_2B_2C_2$ which are inscribed in S_1 and circumscribed about S_2. Let S be the conic w.r.t. which both triangles are self conjugate, and reciprocate the system w.r.t. S.

Then the triangles $A_1B_1C_1$, $A_2B_2C_2$ being self conjugate w.r.t. S reciprocate into themselves, and therefore S_1 reciprocates into the conic touching the six sides of these triangles i.e. into S_2.

∴ S is a conic w.r.t. which S_1 and S_2 are reciprocal. Q.E.D.

Corollary. If a triangle ABC is inscribed in a conic S_1 and circumscribed about a conic S_2, and if PQR is the common self-conjugate triangle of S_1 and S_2 then A, B, C, P, Q, R lie on a conic.

This follows from Theorem 135 and Theorem 78, Corollary, since ABC and PQR are each self conjugate w.r.t. the conic S.

EXERCISE XII. b.

1. If the polar circle of a triangle circumscribing a conic passes through the centre of the conic, prove that the conic must be a rectangular hyperbola.

2. PQR is a triangle self-conjugate w.r.t. a rectangular hyperbola, centre C; prove that P, Q, R, C lie on a circle.

3. The sides of a triangle touch a conic S_1 at A, B, C, and its vertices lie on the conic S_2; PQR is the common self-conjugate triangle of S_1, S_2; prove that A, B, C, P, Q, R lie on a conic.

4 PQR is a triangle self conjugate w r t a conic, centre C , prove that the asymptotes of any hyperbola through P, Q, R, C are parallel to conjugate diameters of the given conic

5 T is the pole of a chord PQ of a conic S_1 , S_2 is a conic through T touching PQ at P , prove that S_1 is harmonically inscribed in S_2

6 CP, CD are conjugate semi diameters of a conic S_1 , S_2 is a hyperbola through C with its asymptotes parallel to CP, CD , prove that S_1 is harmonically inscribed in S_2

7 AB is the diameter of a circle S_1 S_2 is a conic having AB as directrix and its corresponding focus on S_1 , prove that S_2 is harmonically inscribed in S_1

8 The centre of a circle S_1 lies on a rectangular hyperbola S_2 , prove that S_1 is harmonically inscribed in S_2

9 Prove that any circle whose centre lies on the directrix of a parabola is harmonically circumscribed about the parabola

10 Prove that any circle through the focus of a parabola circumscribes triangles in which the parabola is inscribed

11 The orthocentre H of a triangle ABC inscribed in a conic S lies on the director circle of S prove that the polar of H w r t S touches the circle w r t which ABC is self conjugate

12 PQ is a chord of a rectangular hyperbola, centre C prove that the pole of PQ lies on the circle through C, which touches PQ at P

13 T is the pole of a chord PQ of a conic S ABC is a self conjugate triangle w r t S prove that any conic through A B C, T cuts PQ harmonically

14 The focus S of a conic σ_1 lies on the director circle of a conic σ_2, if σ_1 is harmonically inscribed to σ_2 prove that σ_1 touches the polar of S w r t σ_2

Theorem 140. If PQR is a triangle self conjugate w r t a conic Σ, there exists an unlimited number of quadrilaterals which circumscribe Σ and have PQR as diagonal line triangle

Project Σ into a circle σ having the projection p of P as centre then qr becomes the line at infinity and pq pr become perpendicular radii of the circle

Suppose *any* tangent to the circle cuts pq, pr at a, b , then from symmetry ab is the side of a rhombus circumscribing the circle and having pqr as diagonal line triangle , therefore an unlimited number of such rhombuses exist Therefore in the original figure an unlimited number of quadrilaterals satisfying the required conditions exist Q E.D.

Theorem 141. [Gaskin s Theorem]

The circumcircle of a triangle, self conjugate w r t a conic, is orthogonal to the director circle of the conic

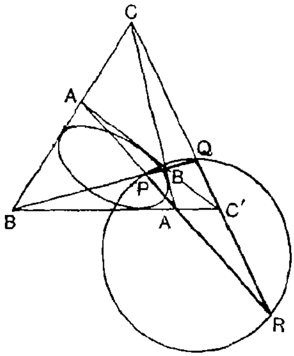

FIG 109

Let PQR be the self conjugate triangle circumscribe a quadri lateral about the conic having PQR as diagonal line triangle, let A, A , B B , C C be the pairs of opposite vertices

Since {AA PR {BB′, PQ' {CC QR are harmonic, the circles whose diameters are AA BB , CC are orthogonal to the circle PQR

Therefore the circle PQR is orthogonal to every circle coaxal with these circles, and is therefore orthogonal to the director circle of the conic Q E D

Note This theorem was discovered by Gaskin in 1852 and independently by Faure in 1860

Theorem 142. The polar circle S of a triangle circumscribing a conic σ is orthogonal to the director circle of σ

By hypothesis σ is harmonically inscribed in S

S is harmonically circumscribed about σ, and so S circumscribes a triangle self conjugate to σ

.. S is orthogonal to the director circle of σ Q E D

Theorem 143. [Steiner's Theorem] The orthocentre of a triangle circumscribing a parabola lies on the directrix

The director circle of a parabola is the directrix, therefore by Theorem 142 the polar circle of the given triangle is orthogonal to the directrix of the parabola, and so its centre lies on the directrix. But its centre is the orthocentre of the triangle, ∴ the orthocentre of the triangle lies on the directrix Q E D

Note. For another method of proof, see Ex XI e, No 15

EXERCISE XII. c.

1. If a variable conic touches a fixed line a and has a fixed self-conjugate triangle pqr, then there are three other fixed lines b, c, d which touch the conic, and pqr is the diagonal line-triangle of the quadrilateral $abcd$.

Further, every conic touching a, b, c, d has pqr as a self-conjugate triangle.

2. Enunciate and prove the dual of No 1

3 Prove that the circumcentre of a triangle, self-conjugate to a parabola, lies on the directrix

4 Prove that the circumcircle of a triangle, self-conjugate to a rectangular hyperbola, passes through the centre of the hyperbola

5 If two circles are harmonically circumscribed about a conic prove that their radical axis passes through the centre of the conic

6 S is a focus of a conic σ inscribed in a triangle PQR, a rectangular hyperbola is drawn through S having PQR as a self conjugate triangle, prove that it touches the major axis of σ

7 Two conics have double contact at A B and are each harmonically circumscribed to a conic S prove that AB touches S

8 The circumcircle of a triangle self conjugate w r t a given conic, is of fixed size find the locus of its centre

9 P is any point on the directrix of a conic, focus S, PT is the tangent from P to the director circle, prove that PT = PS

10. A rectangular hyperbola is harmonically circumscribed to a parabola prove that the axis of the parabola is parallel to the polar of the focus of the parabola w r t the hyperbola

11 PQR is a variable triangle self conjugate w r t a given parabola, if P is fixed, prove that the circle PQR belongs to a fixed coaxal system.

12 a, b are the axes of a variable conic inscribed in a given triangle, find the centre of the conic for which a^2+b^2 is least

13. If a triangle is self-conjugate w.r.t. a rectangular hyperbola, prove that its in-centre and ex-centre lie on the curve.

14. PT, QT are tangents to a parabola ; P is the point of contact of PT, and PQ is drawn perpendicular to TQ and meets the directrix at D ; prove that $\angle$DTP $=90°$.

15. A conic is inscribed in a triangle self-conjugate w r t a rectangular hyperbola and has one focus at the centre of the hyperbola ; prove that it is a parabola.

16. G is the centroid of a triangle ABC circumscribing a parabola; prove that the polar of G w r t. the parabola touches the conic which passes through A, B, C and has its centre at G.

17. The polar circle of a triangle circumscribing a conic passes through a focus ; prove that the orthocentre of the triangle lies on a directrix

18. A variable conic is inscribed in a fixed triangle ; if the sum of the squares of its axes is constant, prove that the locus of its centre is a circle whose centre is the orthocentre of the triangle.

19. Two circles S_1, S_2 are harmonically circumscribed about a conic σ ; prove that any circle coaxal with S_1, S_2 is harmonically circumscribed about σ.

Generalise this theorem.

20. Two circles S_1, S_2 are harmonically circumscribed about a conic σ ; prove that the limiting points of the coaxal system defined by S_1, S_2 lie on the director circle of σ.

21. BE, CF are altitudes of the triangle ABC ; EF meets BC at H ; prove that the focus of the parabola inscribed in ABC and touching EF lies on AH.

22. Find the locus of the circumcentre of the triangle formed by two fixed tangents and one variable tangent of a parabola.

23. σ_1 is a parabola harmonically inscribed in a hyperbola σ_2 , prove that the asymptotes of σ_2 are conjugate lines w r t σ_1.

24. A system of conics touch three fixed lines ; prove that their director circles have a common radical centre.

25. S, H are the foci of a conic Σ inscribed in $\triangle$ABC, σ_1, σ_2 are the conics inscribed in $\triangle$ABC having S, H as centres If r_1, r_2 are the radii of the director circles of σ_1, σ_2, and if c is the major axis of Σ, prove that $r_1^2+r_2^2=c^2$.

Theorem 144. (*1*) Pairs of points P, P′, Q, Q′, R, R′ divide harmonically the joins of opposite vertices A, A′, B, B′, C, C′ of a quadrilateral then the six points P, P′, Q, Q′, R, R′ lie on a conic

(2) [Hesse's Theorem] If two pairs of opposite vertices of a quadrilateral are conjugate points w r t a conic Σ, then the third pair of opposite vertices is also conjugate w r t Σ

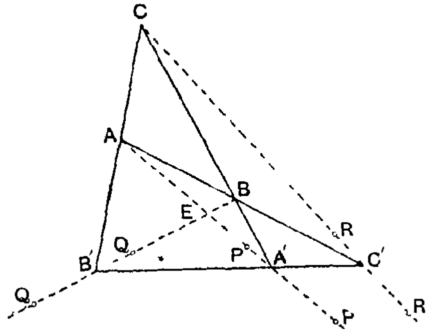

FIG 110

(1) Project R R′ into the circular points at infinity, then ABA′B′ becomes a rectangle since B{AA′, RR′} is harmonic

In the projected figure, since $\{aa', pp'\}$ and $\{bb', qq'\}$ are harmonic, $ep \cdot ep' = ea^2 = eb^2 = eq \cdot eq'$

$\therefore$ p, p', q, q' lie on a circle

$\therefore$ in the original figure P, P′ Q Q′ R R′ lie on a conic

(2) Let A, A′ and B, B′ be conjugate w r t Σ, and let Σ cut AA′, BB′, CC′ at P, P′, Q, Q′, R, R′

By hypothesis {AA′, PP′} and {BB′, QQ′} are harmonic · but only one conic can be drawn through P, P′, Q, Q′, R, therefore, by (1), {CC′, RR′} is harmonic

$\therefore$ C, C′ are conjugate w r t Σ. Q E D

Corollary. If two pairs of opposite sides of a quadrangle are conjugate lines w r t a conic Σ, then the third pair of opposite sides are also conjugate w r t Σ.

This is the dual of (2).

Theorem 145. If two conics S_1, S_2 are harmonically circumscrib to a conic Σ, then every conic of the pencil determined by S_1, S_2 also harmonically circumscribed to Σ.

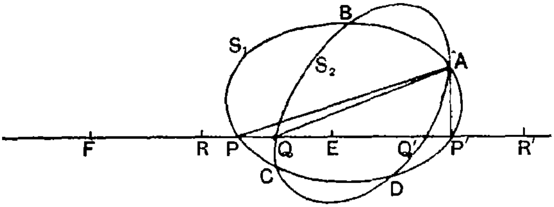

FIG 111

Let S_1, S_2 meet at A, B, C, D ; let the polar of A w r t Σ cut ' at P, P' and S_2 at Q, Q' and Σ at E, F, and let it cut any other con S_3 of the pencil at R, R'

By hypothesis, APP', AQQ' are self conjugate triangles w r t Σ therefore {PP' , EF} and {QQ' , EF} are harmonic

E, F are the double points of the involution defined by P, P Q, Q' But, by Desargues' theorem, R R is a point pair of th involution

. {RR' , EF} is harmonic

ARR' is a self conjugate triangle w r t Σ

$\therefore$ S_3 is harmonically circumscribed about Σ Q E D

Corollary. If two conics S_1, S_2 are harmonically inscribed in conic Σ, then every conic of the range determined by S_1, S_2 is als harmonically inscribed in Σ

This is the dual theorem

Note. Theorem 144, Corollary, arises as a special case of Theoren 145 when S_1, S_2 are two line pairs similarly, Theorem 144 (2) is special case of Theorem 145, Corollary.

Theorem 146. If the sides of a triangle PQR touch a conic S_1, and if Q, R move on conics S_2, S_3 confocal to S_1, then P moves on a conic S_4, also confocal to S_1.

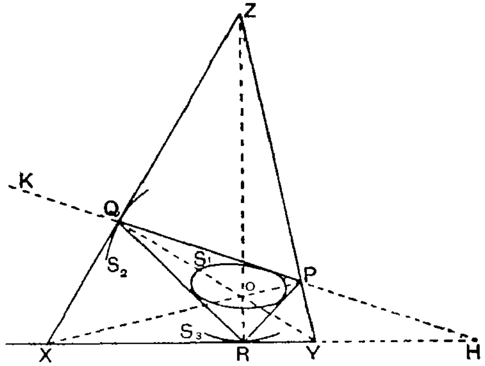

FIG 112

Let P′Q′R′ be an adjacent position of PQR.

By Theorem 134, P, P′, Q, Q′, R, R′ lie on a conic σ. Produce PP′, QQ′, RR′ *to form the triangle* XYZ, *then, in the limit,* σ *touches* the sides of XYZ at P, Q, R, and by Brianchon's theorem PX, QY, RZ are concurrent, at O say.

Produce QP to meet XY at H and produce PQ to K.

From the quadrilateral QPYX, {XY, RH} is harmonic.

∴ Q{XY, RK} = Q{XY, RH} is harmonic.

But by Theorem 128, since ZQX touches a conic S_2 confocal to S_1, ∠XQR = ∠ZQP = ∠XQK,

$$\therefore \angle XQY = 90^\circ.$$

$$\therefore \text{QY is perpendicular to XZ.}$$

Similarly, since R moves on a conic S_3 confocal to S_1, RZ is perpendicular to XY, therefore O is the orthocentre of the triangle XYZ;

$$\therefore \text{PX is perpendicular to YZ.}$$

∴ by similar reasoning, P moves on a conic confocal to S_1.

Q.E.D

Note. This theorem is due to Chasles.

EXERCISE XII. d.

1. ABC, PQR are two triangles inscribed in a conic; two conics drawn circumscribing ABC, PQR respectively and having double cont with each other; prove that their chord of contact touches the co w.r.t. which ABC, PQR are self-conjugate.

2. S_1, S_2 are two conics with parallel asymptotes and are ea harmonically circumscribed to a conic S_3: prove that the comm chord of S_1, S_2 passes through the centre of S_3.

3. If two confocal conics are harmonically inscribed in a third conic prove that S is a rectangular hyperbola.

4. Two parabolas are inscribed in a triangle ABC and are harmon ally inscribed in a conic S, if D is the centre of S, prove that AD, E are conjugate w.r.t. S.

5. If two triangles are conjugate to each other w.r.t. a conic Σ, pro that the meets of corresponding sides are collinear, and that the joins corresponding vertices are concurrent and that the point of concurren is the pole of the line of collinearity w.r.t. Σ. (See Def 3, p. 49.)

6. If a quadrilateral circumscribes a conic S, and if three of it vertices trace out conics confocal to S, then every vertex traces out conic confocal to S, provided that each side passes through at least on of the given vertices.

7. [Poncelet's Theorem.] **If a variable triangle ABC is inscribed in given circle S, and if AB, AC touch fixed circles coaxal with S, then B(touches a circle coaxal with S.**

8. Prove that the cross-ratio of the pencil formed by the polars o any point H w.r.t. four fixed conics having four common points is independent of the position of H.

9. Four conics pass through four common points at A, B, C, D, prove that the cross-ratio of the tangents at A to the conics is equal tc that of the tangents at B.

10. ABCD is a given quadrangle: a variable conic has ABC as a self-conjugate triangle and its director circle passes through D; find the locus of its centre.

11. The conic S_1 is harmonically circumscribed to the conic S_2; if a tangent to S_2 cuts S_1 at A, B, prove that the tangents at A, B to S_1 are conjugate w.r.t. S_2.

12. The orthocentre of a variable triangle circumscribing a given conic is at a focus of the conic; prove that the polar circle and the circumcircle of the triangle are fixed.

13. S is a point on the polar circle of the triangle ABC, prove that a directrix of the conic which is inscribed in ABC and has one focus at S passes through the orthocentre of ABC

14. A variable triangle is inscribed in a fixed circle and circumscribes a fixed parabola, prove that the locus of its centroid is a straight line

15 A variable ellipse is inscribed in a triangle ABC and passes through the circumcentre of ABC, prove that its director circle touches the circumcircle and the nine point circle of ABC

What is the locus of the centre of the ellipse ?

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[*The numbers refer to pages.*]

w.r t = with respect to

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